INVESTIGATION OF THE DEFORMED FERMI SURFACES MECHANISM FOR PAIRING OF TWO SPECIES OF FERMIONS WITH MISMATCHED FERMI SURFACES

A Thesis

by

JIANXU LU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2007

Major Subject: Physics

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Approved by:

Chair of Committee,	Chia-Ren Hu
Committee Members,	Joseph H. Ross, Jr.
	Stephen A. Fulling
Head of Department,	Edward S. Fry

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ABSTRACT

Investigation of the Deformed Fermi Surfaces
for Pairing of Two Species of Fermions
with Mismatched Fermi Surfaces. (December 2007)

Jianxu Lu, B.S., University of Science and Technology of China
Chair of Advisory Committee: Dr. Chia-ren Hu

Variational method is used to investigate, at zero temperature, the deformed-Fermi-surfaces mechanism for solving the problem of superconducting pairing of two species of fermions (i.e., spin-up and -down) of mismatched Fermi surfaces due to the existence of a uniform exchange or Zeeman field. After analyzing the depairing regions in the whole three-dimensional parameter space, we obtain a trial ground-state wave-function as a function of the three variational parameters, one of which is the gap function. Then within the frame work of the weak-coupling BCS theory, the expectation value of the Hamiltonian of a conductor under an exchange or Zeeman field is derived, from which a gap equation is derived by differentiation. The influence of deformed Fermi surfaces on the chemical potential is then calculated. Computer programing is finally used to solve the gap equation, and find the minimum-energy state with respect to the remaining two variational parameters ($\delta\mu$ and z). These two parameters are better than the original parameters used in the trial Hamiltonian when compared with the FF state. And we also found if we keep the total number of electrons fixed, the system prefers an unchanged chemical potential and the ground

state energy of the deformed-Fermi-surfaces state, which is found to be an angle dependent case of Sarma's solution III, is no better than that of the unpolarized BCS state.

To the people who care for me and who I care for

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CHAPTER I

INTRODUCTION

A. The Structure of This Thesis

In chapter I, we will go over some important results of the BCS theory, Sarma's paper and FF(Fulde and Ferrell) state, and give our reasons why we need to investigate the DFS(Deformed Fermi Surface) states for superconductor. In chapter II, we will find depairing(or blocking) regions of the the DFS state. In chapter III, we use a trial ground state wave function to investigate the chemical potential and minimize the original Hamiltonian of S-wave superconductor. And some computer programs used in Chapter III are attached to the appendix. Chapter IV is the conclusion.

B. BCS Theory

In 1957, John Bardeen, Leon N. Cooper, and J. Robert Schrieffer [1], published the now famous BCS theory, and Found a electron with momentum $\hbar\vec{k}$ and spin $\uparrow$ and a electron with momentum $-\hbar\vec{k}$ and spin $\downarrow$ near their common Fermi surfaces to pair up into what is now known as a Cooper pair. When their energies, relative to the Fermi energy, are smaller than a cut-off energy $\hbar\omega_D$, a process of virtual exchanges of phonons between them will lead to an attractive interaction. Within weak-coupling limits, i.e., $N(0)V << 1$ where $N(0)$ is the electron density of states on the Fermi surface, and $V > 0$ is an effective coupling constant representing the

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The journal model is *Physical Review A* phonon-mediated attractive electron-electron interaction, this interaction results in a many-body gap. At zero temperature, we have the gap:

$$\Delta_0 = 2\hbar\omega_D \exp\{-\frac{1}{N(0)V}\}\,, \tag{1.1}$$

and the energy relative to the normal-state energy:

$$E_{BCS} = -\frac{1}{2}N(0)\Delta_0^2\,. \tag{1.2}$$

(In this model, V is assumed to be a constant within $\pm\hbar\omega_D$ of the Fermi energy due to the smallness of the ratio of $\hbar\omega_D$ to the Fermi energy. Here ω_D is assumed to be just the theoretical cut-off frequency in the Debye model of phonons in a conductor, which is well known as the Debye frequency. Also, at low temperatures other than zero, the BCS theory successfully explained the phenomenon of superconductivity observed ubiquitously in many metals and alloys and has also been widely used to explain all kinds of phenomena related to superconductivity.

C. Clogston-Chandrasekhar Limit

The influence exerted by an exchange field on a superconductor is our concern. In 1962, B. S. Chanderasekhar [2] and A. M. Clogston [3] predicted a natural upper limit to the critical field of a superconductor, which is usually called the Clogston-Chandrasekhar limit. Clogston argued in his paper that even if the superconducting BCS state could be realized without the orbital magnetic screening known as the Meissner effect, and only the Zeeman energy associated with the spins is considered, a critical field should still exist for the system to become normal. At this Clogston-

Chandrasekhar limit field H_0, the free energy of the superconducting state, F_S, should be $\frac{1}{2}\chi_p H_0^2$ below the zero-field normal-state free-energy F_N, where χ_p is the Pauli susceptibility, because electrons in the normal state in an external magnetic field shows Pauli paramagnetism, with corresponding lowering of its free energy, whereas χ_p is reduced to zero in the superconducting state, due to the opening of a gap at the Fermi surface. On the other hand, a simple argument based on Fermi statistics gives the Pauli susceptibility $\chi_p = 2N(0)\mu_B^2$, where μ_B is the Bohr magneton. Thus

$$F_N - N(0)\mu_B^2 H_0^2 = F_S \,. \tag{1.3}$$

But from the BCS theory, we have:

$$F_N - \frac{1}{2}N(0)\Delta_0^2 = F_S \,. \tag{1.4}$$

Thus he obtained:

$$\mu_B H_0 = \frac{1}{\sqrt{2}}\Delta_0 \,. \tag{1.5}$$

In 1963, within the frame work of the BCS Theory, G. Sarma [4] discussed the effects of a uniform exchange field acting on the conduction electrons in a superconductor and verified this limit. His theory begins with adding the Zeeman energy to the reduced Hamiltonian of the BCS theory, which includes interaction matrix elements between pairs of electrons of zero total momentum:

$$\mathscr{H} = \sum_{\vec{k}}(\epsilon'_k + h)c^\dagger_{\vec{k}\uparrow}c_{\vec{k}\uparrow} + \sum_{\vec{k}}(\epsilon'_k - h)c^\dagger_{-\vec{k}\downarrow}c_{-\vec{k}\downarrow} - \sum_{\vec{k},\vec{k}'} V_{\vec{k},\vec{k}'}c^\dagger_{\vec{k}'\uparrow}c^\dagger_{-\vec{k}'\downarrow}c_{-\vec{k}\downarrow}c_{\vec{k}\uparrow} \,,$$

where $\epsilon'_k \equiv \frac{\hbar^2 k^2}{2m} - \mu'$ is the energy of an electron of momentum $\hbar\vec{k}$ measured relative to the chemical potential μ'. This Hamiltonian reduces to the corresponding one in the BCS theory if $h \equiv \mu_B H$ is set to zero.

As in the BCS theory, which is for an s-wave superconductor, Sarma assumed:

$$V_{kk'} = \begin{cases} V & \text{if } |\epsilon_k| < \hbar\omega_D \,, \\ 0 & \text{otherwise} \,. \end{cases} \tag{1.6}$$

By using mean-field approximation, the Bogliubov-Valatin transformation, and self-consistency condition, he then finds the finite-temperature gap equation and free energy. At zero temperature three solutions are obtained:

Solution I is the Pauli paramagnetism of the normal state with the gap equal to 0, and the energy equal to:

$$E_1 = \sum_{\vec{k}} \epsilon'_k - \sum_{|\epsilon_k|>h} |\epsilon'_k| - \sum_{|\epsilon'_k|<h} h \,. \tag{1.7}$$

Solution II is an unpolarized BCS ground state: the energy of this state is h independent, and is just E_{BCS}.

Solution III is a superconducting state with a depairing region $[-\sqrt{h^2-\Delta_3^2} < \epsilon(k) < \sqrt{h^2-\Delta_3^2}]$, Where Δ_3 is just Δ obtained in this solution (and E_3 its energy at $T = 0$), and its gap equation and energy are found to be:

$$\frac{1}{N(0)V} = \int_{\sqrt{h^2-\Delta_3^2}}^{\hbar\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2+\Delta_3^2}} \,, \tag{1.8}$$

$$E_3 = \sum_{\vec{k}} \epsilon'_k - \sum_{|\epsilon'_k|<\sqrt{h^2-\Delta^2}|} h - \sum_{|\epsilon'_k|>\sqrt{h^2-\Delta^2}} \sqrt{\epsilon'^2_k + \Delta_3^2} + \frac{\Delta_3^2}{V} \,. \tag{1.9}$$

We can easily derive the normal-state energy from any one of the three energy expressions by setting the gap and h equal to zero, which gives:

$$E_n = \sum_{\vec{k}} \epsilon'_k - \sum_{\vec{k}} |\epsilon'_k| \,. \tag{1.10}$$

Then these gaps and energy differences $E_n - E_1$, $E_n - E_{BCS}$, and $E_n - E_3$ can be plotted versus the magnetic field h:

From fig 1, we can see that Δ_3 and E_3 exist only when $h \in [\frac{\Delta_0}{2}, \Delta_0]$. And from fig 2, we can see that E_3 is not better than E_{BCS} or E_1, the energy of the normal state with Pauli paramagnetism. So solution III, a superconducting state with depairing region, will never occur in nature. Thus according to the Sarma theory, when the strength of the exchange field is below the Clogston-Chandrasekhar limit, we always have the unpolarized BCS state; and above this limit, the normal state with Pauli paramagnetism will prevail.

D. The FFLO State

Soon after Sarma's paper, Fulde and Ferrell (FF) [5] in the US, and Larkin and Ovchinnikov (LO) [6] in the USSR, proposed independently and contemporarily two slightly different versions of a new superconducting state. These two versions of the new superconducting state are later known collectively as the FFLO state (or LOFF) state. In FF's paper, they showed that the transition from the BCS state to the FF state occurs at a field strength less than the Clogston-Chandrasekhar limit. (Cui [7] later showed that it is at $h \simeq 0.704\Delta_0$ for the s-wave case.)

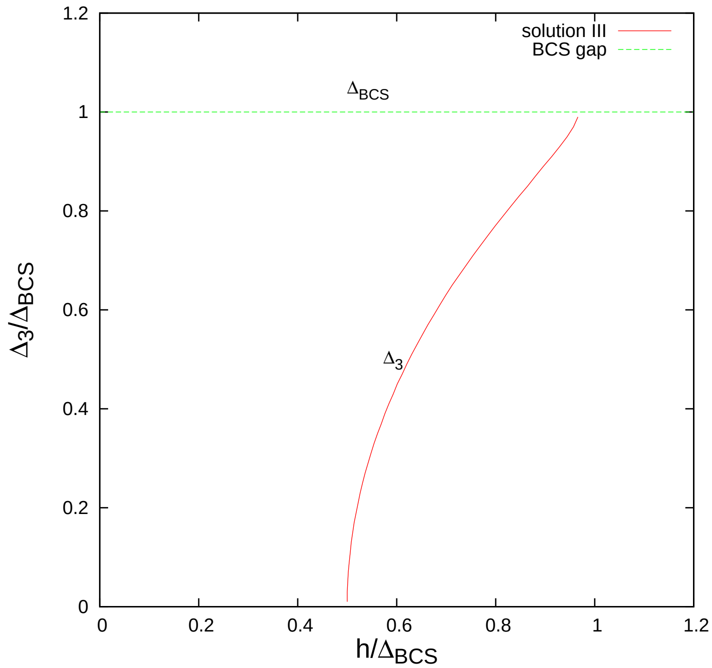

Fig. 1. Gap of Sarma's solution III with N(0)V=0.3, $\beta_T = \frac{1}{k_B T} = 1000/\Delta_{BCS}$

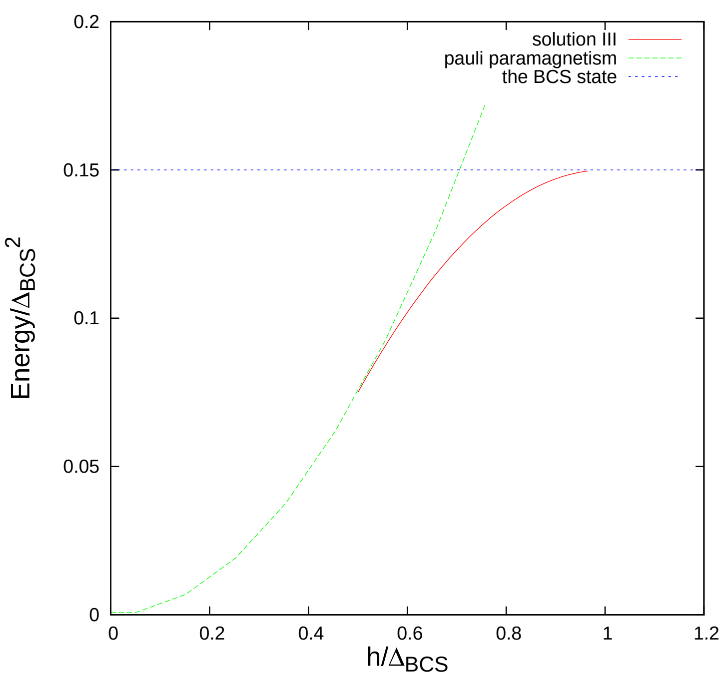

Fig. 2. Energy of Sarma's solution III with N(0)V=0.3, $\beta_T = \frac{1}{k_B T} = 1000/\Delta_{BCS}$

The reduced Hamiltonian is now taken to be:

$$\mathscr{H} = \sum_{\vec{k}\uparrow}(\epsilon'_k + h)c^\dagger_{k\uparrow}c_{k\uparrow} + \sum_{-\vec{k}\downarrow}(\epsilon'_k - h)c^\dagger_{-k\downarrow}c_{-k\downarrow} - \sum_{\vec{k},\vec{k\prime},\vec{q}} V_{k,k\prime}c^\dagger_{\vec{k\prime}+\vec{q}\uparrow}c^\dagger_{-\vec{k\prime}+\vec{q}\downarrow}c_{-\vec{k}\downarrow+\vec{q}}c_{\vec{k}\uparrow+q}\,,$$

which emphasizes interaction of pairs of electrons of combined momentum $2\hbar\vec{q} \neq 0$.

Under mean-field approximation and the Bogoliuobv-Valatin transformation:

$$\begin{pmatrix} c_{k+q\uparrow} \\ c^\dagger_{-k+q\downarrow} \end{pmatrix} = \begin{pmatrix} u_k* & v_k \\ -v_k* & u_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \beta^\dagger_{k\downarrow} \end{pmatrix}\,, \tag{1.11}$$

it can be written as:

$$\mathscr{H}_{MF} = \sum_{\vec{k}}(E_{k+}\alpha^\dagger_{k\uparrow}\alpha_{k\uparrow} + E_{k-}\beta^\dagger_{k\downarrow}\beta_{k\downarrow}) + \text{const.}\,, \tag{1.12}$$

where

$$u_k = \Big[\frac{1}{2}\Big(1 + \frac{\epsilon^{(s)}_k}{E_k}\Big)\Big]^{1/2}\,, \tag{1.13}$$

$$v_k = \Big[\frac{1}{2}\Big(1 - \frac{\epsilon^{(s)}_k}{E_k}\Big)\Big]^{1/2}\,, \tag{1.14}$$

$$E_{k\pm} = E_k \pm \epsilon^{(a)}_k\,, \tag{1.15}$$

$$E_k = \sqrt{\epsilon^{(s)2}_k + \Delta^2_q}\,, \tag{1.16}$$

$$\epsilon^{(s)}_k = \frac{1}{2}(\epsilon'_{k+q\uparrow} + \epsilon'_{-k+q\downarrow}) = \frac{\hbar^2 k^2}{2m} - \mu' + \frac{\hbar^2 q^2}{2m}\,, \tag{1.17}$$

$$\epsilon^{(a)}_k = \frac{1}{2}(\epsilon'_{k+q\uparrow} - \epsilon'_{-k+q\downarrow}) \simeq \frac{\hbar^2 k_F}{m} q\cos\theta\,, \tag{1.18}$$

and

$$\Delta_q = -\sum_k V_{kk'}\langle c_{-\vec{k}\downarrow+\vec{q}}c_{\vec{k}\uparrow+q}\rangle\,. \tag{1.19}$$

Here θ is the angle between $\vec{k}$ and $\vec{q}$. Let $Q = \frac{\hbar^2 k_F q}{m} = \hbar v_F q$, the depairing regions are determined by $E_{k+} < 0$, which gives:

$$-1 \leq \cos\theta \leq \frac{h - \Delta_q}{Q} \,,$$
$$-\sqrt{(Q\cos\theta - h)^2 - \Delta_q^2} \leq \epsilon_k^s \leq \sqrt{(Q\cos\theta - h)^2 - \Delta_q^2} \,, \tag{1.20}$$

and $E_{k-} < 0$, which gives:

$$\frac{h + \Delta_q}{Q} \leq \cos\theta \leq 1 \,,$$
$$-\sqrt{(Q\cos\theta - h)^2 - \Delta_q^2} \leq \epsilon_k^s \leq \sqrt{(Q\cos\theta - h)^2 - \Delta_q^2} \,. \tag{1.21}$$

From equation (Eq. (1.20)) and (Eq. (1.20)), we can find that the depairing regions are along $\pm\vec{q}$, and Takada and Izuyama [8] have given rough sketches of them. When $\vec{q}$ vanishes, this solution will reduce to Sarma's solution III. Its gap equation for a three-dimensional system has been given by Shimahara [9]:

$$\begin{aligned} ln\frac{\Delta_0}{\Delta_q} = \frac{1}{2}\int_{\frac{h+\Delta_q}{Q}}^{1} ln\frac{|Q\cos\theta - h| + \sqrt{(Q\cos\theta - h)^2 - \Delta_q^2}}{\Delta_q} d\cos\theta \\ + \frac{1}{2}\int_{-1}^{\frac{h-\Delta_q}{Q}} ln\frac{|Q\cos\theta - h| + \sqrt{(Q\cos\theta - h)^2 - \Delta_q^2}}{\Delta_q} d\cos\theta \,. \end{aligned} \tag{1.22}$$

so far we have reviewed the FF state only. the LO state is a bit more complicated in that it allows pairs of both $\pm 2\hbar\vec{q}$ momenta, and possibly their higher harmonics, leading to a real periodic gap-function order parameter that is not limited to being sinusoidal, and no net supercurrent in the system. In summary, although there are depairing regions, all different versions of the FFLO state have spatially varying gap function in its phase and/or magnitude, electron pairs of non-zero momentum(a), and a free energy lower than those of the paramagnetic (i.e., polarized) normal state and the unpolarized BCS state around the Clogston-Chandrasekhar limit.

This is a new state that can exist between the BCS state and the normal state. A number of theorists have since worked on this FFLO state and recent experiments on quasi two dimensional superconductors such as $CeCoIn_5$ have shown some signatures of this state [10, 11, 12, 13, 14, 15, 16], but up to the present time, it is not yet clear whether this state has been observed, or what has been observed between the superconducting state and the normal state at low temperatures and high magnetic fields might be some other yet unknown state.

E. Why Deformed Fermi Surfaces?

The success of the FFLO state is due to the electron pairs' momentum $2q$, or, in other words, due to the fact that the pairing is now between a $(\vec{k}+\vec{q},\uparrow)$ electron and a $(-\vec{k}+\vec{q},\downarrow)$ electron, which is like first shifting both Fermi surfaces by $-\vec{q}$, and then do the usual $\vec{k}\uparrow$ and $-\vec{k}\downarrow$ pairing. However, for any fixed $\vec{q}$, this scheme can put both electrons on their respective Fermi surfaces only for some parts of their Fermi surfaces corresponding to $\vec{k}$ away from the directions of $\pm\vec{q}$. A close look at the depairing region and gap equation of Sarma's solution III and Takada's result shows that the existence of a finite $\vec{q}$ decreases the size of the depairing region.

Recently, as an alternative approach to attack this problem, the Deformed-Fermi-Surface (DFS) pairing scheme were proposed by H. M*ü*ther and A. Sedrakian [17, 18], who are actually working in nuclear physics. This new state seems to be able to compete with the FFLO state. The essential idea of this scheme is to deform both

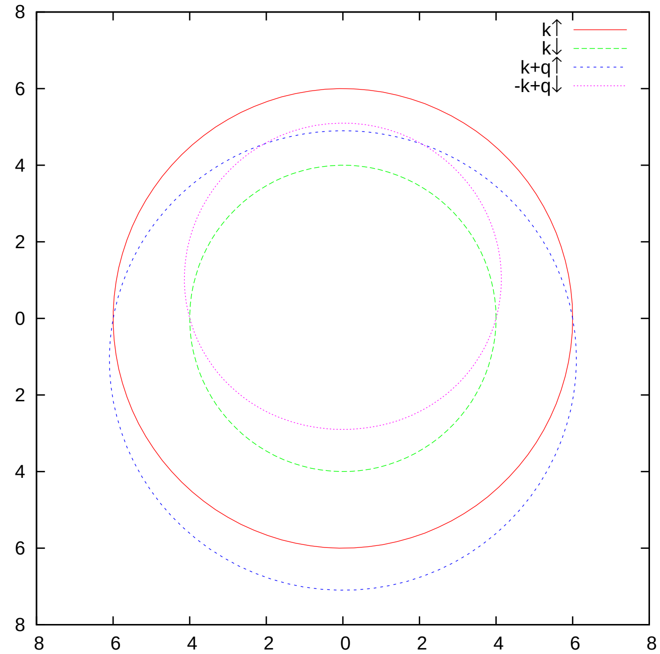

Fig. 3. A rough picture of fermi surfaces of FF states

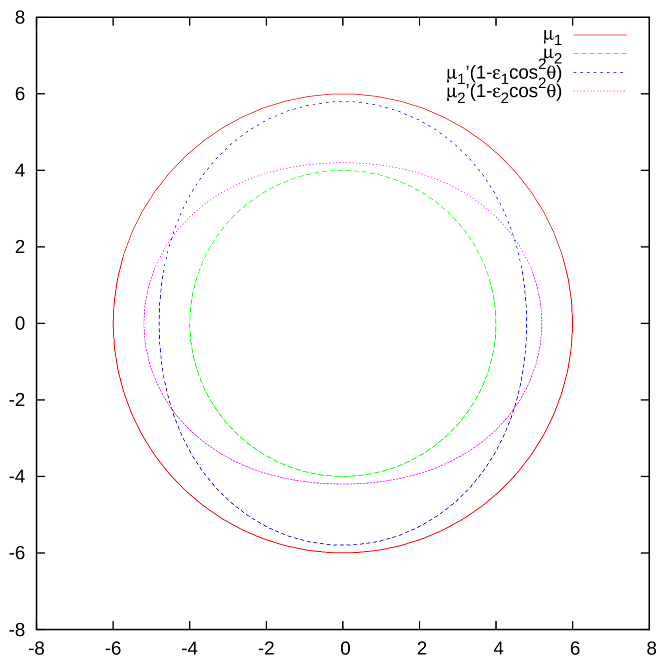

Fig. 4. A rough picture of fermi surfaces of DFS states

Fermi surfaces of spin-up and spin-down electrons, as shown in fig. 3, to make parts of these two Fermi surfaces to at least nearly match each other, so that the usual kind of Cooper pairs can be formed in these regions to lower the total free energy of the system. The region that can pair in this DFS case, as shown in fig. 4 seems to be larger than that of the FFLO state. If the DFS pairing scheme can really improve the free energy, then It will be another candidate state for interpreting those new phenomena observed in the experiments. And they have claimed the DFS state "has lower free energy than the normal, BCS, and LOFF states in a wider range of asymmetries,

M*ü*ther and Sedrakian found that the DFS pairing state has favorable energy in the parameter region defined by 0.03$\leq \alpha \leq$0.06 and 0.12$\leq \delta\epsilon \leq$.16", where

$$\alpha \equiv \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\,, \tag{1.23}$$

with ρ_1 and ρ_2 the densities of two species of particles, and

$$\delta\epsilon \equiv \frac{\epsilon_1 - \epsilon_2}{2}\,, \tag{1.24}$$

with ϵ_1 and ϵ_2 the ratios of the deformation energies to the Fermi energies respectively, assuming that the deformations all have a $\cos^2\theta$ dependence with θ the polar angle of a point on a three-dimensional Fermi surface. In a conductor, ρ_1 and ρ_2 can be expressed in numbers of two spin species of electrons, and $\rho_1 - \rho_2$ in the difference of numbers of two spin species of electrons. We thus should have:

$$\rho_1 - \rho_2 \sim 2\frac{N(0)}{\Omega}h\,, \tag{1.25}$$

and

$$\rho_1 + \rho_2 \sim \frac{N(0)}{\Omega} \bar{E}_F \,, \tag{1.26}$$

where Ω is the volume, and E_F the mean Fermi energy of the electrons. However, the Zeeman energy for practically applicable magnetic fields is very small when compared with $\hbar\omega_D$, let alone the Fermi energy or the chemical potential in a conductor. Thus 0.03 as the lower limit of α appears to be a impractical requirement. Also, a ratio of $\delta\epsilon > 0.1$ will make the deformation energy lager than $\hbar\omega_D$, which also seems to be impractically large. In this work we wish to investigate whether the DFS pairing state is indeed a competitive state or not in a conductor with mismatched Fermi surfaces.

CHAPTER II

THE TRIAL HAMILTONIAN AND DEPAIRING REGIONS

A. Trial Hamiltonian

We start with the trial Hamiltonian defined by:

$$\begin{aligned}\mathscr{H}_{trial} &= \sum_{\vec{k}\uparrow}[\frac{\hbar^2 k^2}{2m} - \mu_1(1-\epsilon_1\cos^2\theta)]c^\dagger_{k\uparrow}c_{k\uparrow} \\ &+ \sum_{-\vec{k}\downarrow}[\frac{\hbar^2 k^2}{2m} - \mu_2(1-\epsilon_2\cos^2\theta)]c^\dagger_{-k\downarrow}c_{-k\downarrow} - \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}c_{-k\downarrow}c_{k\uparrow}, \end{aligned} \tag{2.1}$$

where μ_1 and μ_2 are trial Fermi energies for electrons of spin-up and spin-down, respectively; ϵ_1 and ϵ_2 are deformation coefficients; θ is the angle between $\vec{k}$ and the axis of symmetry breaking, coupling cosntant $V_{k,k\prime}$ defined by Eq. 1.6 for S-wave and the mean chemical potential μ is given by:

$$\mu \equiv \frac{\mu_1+\mu_2}{2}\,. \tag{2.2}$$

We also define

$$\mu \equiv \frac{\hbar^2 k_F^2}{2m_1}\,, \tag{2.3}$$

$$\delta\mu \equiv \frac{\mu_1-\mu_2}{2}\,. \tag{2.4}$$

By using self-consistent mean field theory, the fluctuations of $c_{-k\downarrow}c_{k\uparrow}$ and $c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}$ are assume to be very small when compared with their expectation values. $\mathscr{H}_{trial}$

can thus be written as:

$$\begin{aligned}\mathscr{H}_{trial} \;\simeq\;& \sum_{\vec{k}\uparrow}[\frac{\hbar^2 k^2}{2m_1} - \mu_1(1-\epsilon_1\cos^2\theta)]c^\dagger_{k\uparrow}c_{k\uparrow} \\ &+\sum_{-\vec{k}\downarrow}[\frac{\hbar^2 k^2}{2m_2} - \mu_2(1-\epsilon_2\cos^2\theta)]c^\dagger_{-k\downarrow}c_{-k\downarrow} - \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}\langle c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}\rangle c_{-k\downarrow}c_{k\uparrow} \\ &-\sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}\langle c_{-k\downarrow}c_{k\uparrow}\rangle \\ &+\sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}\langle c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}\rangle\langle c_{-k\downarrow}c_{k\uparrow}\rangle \,. \end{aligned} \tag{2.5}$$

To simplify calculation, let

$$\Delta_{k\prime} = \sum_{\vec{k}} V_{k,k\prime}\langle c_{-k\downarrow}c_{k\uparrow}\rangle\,, \tag{2.6}$$

$$a_k = \frac{\hbar^2 k^2}{2m_1} - \mu_1(1-\epsilon_1\cos^2\theta)\,, \tag{2.7}$$

$$b_k = \frac{\hbar^2 k^2}{2m_2} - \mu_2(1-\epsilon_2\cos^2\theta)\,, \tag{2.8}$$

$$c = \mu_1(1-\epsilon_1\cos^2\theta)\,, \tag{2.9}$$

$$d = \mu_2(1-\epsilon_2\cos^2\theta)\,, \tag{2.10}$$

$$\alpha(\theta) = \alpha = \frac{\mu_1\epsilon_1 + \mu_2\epsilon_2}{2}\cos^2\theta\,, \tag{2.11}$$

$$z = \frac{\mu_1\epsilon_1 - \mu_2\epsilon_2}{2}\,, \tag{2.12}$$

Then we obtain:

$$\begin{aligned}\mathscr{H}' \;\simeq\;& \sum_{\vec{k}} a_k c^\dagger_{k\uparrow}c_{k\uparrow} + \sum_{-k} b_k c^\dagger_{-k\downarrow}c_{-k\downarrow} \\ &-\sum_{\vec{k}}\Delta c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow} - \sum_{\vec{k}}\Delta c_{-k\downarrow}c_{k\uparrow} + \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}\langle c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}\rangle\langle c_{-k\downarrow}c_{k\uparrow}\rangle\,. \end{aligned} \tag{2.13}$$

B. Bogoliubov-Valatin Transformation

For an s-wave superconductor, we have:

$$
\begin{aligned}
\mathscr{H}' &\simeq \sum_{\vec{k}\uparrow} a_k c^\dagger_{k\uparrow} c_{k\uparrow} + \sum_{-\vec{k}\downarrow} b_k c^\dagger_{-k\downarrow} c_{-k\downarrow} - \sum_{\vec{k}} \Delta c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - \sum_{\vec{k}} \Delta c_{-k\downarrow} c_{k\uparrow} + \frac{\Delta^2}{V} \\
&= \sum_{\vec{k}} \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \begin{pmatrix} a_k & -\Delta \\ -\Delta & -b_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + \sum_{-\vec{k}\downarrow} b_k + \frac{\Delta^2}{V} . \qquad (2.14)
\end{aligned}
$$

The eigenvalues of the above matrices are

$$
\lambda_{k1,k2} = \frac{a_k - b_k}{2} \pm \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2} \,, \qquad (2.15)
$$

and

$$
\begin{aligned}
u_k^2 &= \frac{1}{2}\left(1 + \frac{\frac{a_k+b_k}{2}}{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2}}\right) & (2.16) \\
v_k^2 &= \frac{1}{2}\left(1 - \frac{\frac{a_k+b_k}{2}}{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2}}\right) & (2.17) \\
\frac{u_k}{v_k} &= \frac{\Delta}{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2} - \frac{a_k-b_k}{2}} = \frac{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2} + \frac{a_k-b_k}{2}}{\Delta} . & (2.18)
\end{aligned}
$$

Assuming that $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\begin{pmatrix} a_3 \\ a_4 \end{pmatrix}$ are the eigenvectors, then we must have:

$$
\begin{pmatrix} a_k - \lambda_{k1} & -\Delta \\ -\Delta & -b_k - \lambda_{k1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \,, \qquad (2.19)
$$

and

$$\begin{pmatrix} a_k - \lambda_{k2} & -\Delta \\ -\Delta & -b_k - \lambda_{k2} \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = 0\,. \tag{2.20}$$

Then we find

$$\frac{a_1}{a_2} = \frac{\Delta}{-\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2} + \frac{a_k-b_k}{2}} = \frac{u_k}{-v_k}\,, \tag{2.21}$$

$$\frac{a_3}{a_4} = \frac{\Delta}{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2} + \frac{a_k-b_k}{2}} = \frac{v_k}{u_k}\,. \tag{2.22}$$

From the above equations, we can see the transformation matrix should be:

$$\mathbf{T} = \begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix}\,, \quad and \quad \mathbf{T}^\dagger \mathbf{T} = I\,, \tag{2.23}$$

$$\mathbf{T}^\dagger \begin{pmatrix} a_k & -\Delta \\ -\Delta & -b_k \end{pmatrix} \mathbf{T} = \mathbf{T}^\dagger \begin{pmatrix} \lambda_{k1} u_k & \lambda_{k2} v_k \\ -\lambda_{k1} v_k & \lambda_{k2} u_k \end{pmatrix} = \begin{pmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{pmatrix}\,, \tag{2.24}$$

so the Hamiltonian is reduced to:

$$\begin{aligned}
\mathscr{H}' &= \sum_{\vec{k}} \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \mathbf{T}\mathbf{T}^\dagger \begin{pmatrix} a_k & -\Delta \\ -\Delta & -b_k \end{pmatrix} \mathbf{T}\mathbf{T}^\dagger \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + \sum_{-\vec{k}\downarrow} b_k + \frac{\Delta^2}{V} \\
&= \sum_{\vec{k}} \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \mathbf{T} \begin{pmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{pmatrix} \mathbf{T}^\dagger \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + \sum_{-\vec{k}\downarrow} b_k + \frac{\Delta^2}{V} \\
&= \sum_{\vec{k}} \begin{pmatrix} \alpha^\dagger_k & \beta_k \end{pmatrix} \begin{pmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta^\dagger_k \end{pmatrix} + \sum_{\vec{k}\downarrow} b_k + \frac{\Delta^2}{V} \\
&= \sum_{\vec{k}} \left(\lambda_{k1}\alpha^\dagger_k \alpha_k + \lambda_{k2}\beta_{k\downarrow}\beta^\dagger_{k\downarrow} \right) + \sum_{-\vec{k}\downarrow} b_k + \frac{\Delta^2}{V}\,, \qquad (2.25)
\end{aligned}$$

where

$$\begin{pmatrix} \alpha_k \\ \beta^\dagger_k \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix}\,, \qquad (2.26)$$

which is well know as Bogliubov-Valatin Transformation.

C. Depairing Regions

Next, we analyze the depairing regions of this trial Hamiltonian. Let

$$E_{k1} \equiv \lambda_{k1} = \frac{a_k - b_k}{2} + \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2}\,, \qquad (2.27)$$

$$E_{k1} \equiv -\lambda_{k2} = -\frac{a_k - b_k}{2} + \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2}\,. \qquad (2.28)$$

For the ground state, when $E_{k1} < 0$, the number operator $\alpha^\dagger_k \alpha_k$ will favor 1 rather than 0, which makes the Hamiltonian looking like it is unreasonable because no quasiparticles should be excited at the ground state. We therefore should use $(1-\alpha_k\alpha_k^\dagger)$ instead of $\alpha_k^\dagger\alpha_k$. Thus we can easily find where depairing occurs. (Some author would rather redefine the annihilation and creation operators [7].) So, we have three cases here:

Case 1:$E_{k1} < 0$ while $E_{k2} > 0$

$$\mathscr{H}' = \sum_{\vec{k}} \left(-E_{k1}\alpha_k\alpha_k^\dagger + E_{k2}\beta_k^\dagger\beta_k\right) + \sum_{\vec{k}\downarrow} b_k + \sum_{\vec{k}}(E_{k1} - E_{k2}) + \frac{\Delta^2}{V}\,; \tag{2.29}$$

Case 2:$E_{k1} > 0$ while $E_{k2} < 0$

$$\mathscr{H}' = \sum_{\vec{k}} \left(E_{k1}\alpha_k^\dagger\alpha_k - E_{k2}\beta_k\beta_k^\dagger\right) + \sum_{-\vec{k}\downarrow} b_k + \frac{\Delta^2}{V}\,;$$

Case3:$E_{k1,k2} > 0$

$$\mathscr{H}' = \sum_{\vec{k}} \left(E_{k1}\alpha_k^\dagger\alpha_k + E_{k2}\beta_k^\dagger\beta_k\right) + \sum_{\vec{k}\downarrow} b_k - \sum_{\vec{k}} E_{k2} + \frac{\Delta^2}{V}\,.$$

Let's first consider the case 1, $E_{k1} < 0$ while $E_{k2} > 0$:

$$\frac{-a_k + b_k}{2} \geq \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2} > 0\,.$$

From $\frac{-a_k+b_k}{2} > 0$, we find:

$$\cos^2\theta < \frac{\delta\mu}{z}\,, \tag{2.30}$$

whereas from $\frac{a_k - b_k}{2} \geq \sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2}$, we find:

$$(\frac{\hbar^2 k^2}{2m} - c)(\frac{\hbar^2 k^2}{2m} - d) + \Delta^2 \leq 0\,. \tag{2.31}$$

This can be solved to obtain:

$$-\alpha(\theta) - \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2} \leq \frac{\hbar^2 k^2}{2m} - \mu \leq -\alpha(\theta) + \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2}\,, \tag{2.32}$$

or

$$\epsilon_- \leq \epsilon_k \leq \epsilon_+\,, \tag{2.33}$$

where

$$\epsilon_\pm = -\alpha(\theta) \pm \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2}\,. \tag{2.34}$$

However,$\epsilon_\pm$ exists only when

$$(\delta\mu - z\cos^2\theta)^2 - \Delta^2 \geq 0\,. \tag{2.35}$$

That is to say;

$$0 \leq \cos^2\theta \leq \frac{\delta\mu - \Delta}{z}\,, \tag{2.36}$$

or

$$\frac{\delta\mu + \Delta}{z} \leq \cos^2\theta \leq 1\,. \tag{2.37}$$

So case 1 exists in the following region:

$$\epsilon_- \leq \epsilon_k \leq \epsilon_+\,, \tag{2.38}$$

and

$$0 \le \cos^2\theta \le \frac{\delta\mu - \Delta}{z} \,. \tag{2.39}$$

let's call this depairing region D_1, or region $D_{1\alpha}$ and $D_{1\beta}$.

The same analysis applies to case 2: $E_{k1} > 0$ while $E_{k2} < 0$, we find it exists only when

$$\epsilon_- \le \epsilon_k \le \epsilon_+ \,, \tag{2.40}$$

and

$$\frac{\delta\mu + \Delta}{z} \le \cos^2\theta \le 1 \,. \tag{2.41}$$

Let's call this depairing region D_2, or region $D_{2\alpha}$ and $D_{2\beta}$.

Here, Some one might ask what happens if $\frac{\delta\mu-\Delta}{z} \le \cos^2\theta \le \frac{\delta\mu}{z}$ and $\epsilon_- \le \epsilon_k \le \epsilon_+$. A second thought tells us that $\epsilon_\pm$ do not exist in this region, i.e. we have $E_{k1,k2} \ge 0$ in this region.

Finally for case 3 $E_{k1,k2} \ge 0$, we have

$$\begin{aligned} &when \quad \frac{a_k - b_k}{2} < 0; \quad -\frac{a_k - b_k}{2} \le \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2} \\ &when \quad \frac{a_k - b_k}{2} > 0; \quad \frac{a_k - b_k}{2} \le \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2} \,, \end{aligned} \tag{2.42}$$

which show us a region complementary to $D_1 + D_2$, let's call this region D, or region D_α and D_β.

In conclusion, in region D, we have no depairing and the Hamiltonian is:

$$\mathscr{H}' = \sum_{\vec{k}} \left(E_{k1}\alpha_k^\dagger \alpha_k + E_{k2}\beta_k^\dagger \beta_k \right) + \sum_{\vec{k}} (b_k - E_{k2}) + \frac{\Delta^2}{V}\,; \tag{2.43}$$

In region D_1, we have some depairing and the Hamiltonian is better written as:

$$\mathscr{H}' = \sum_{\vec{k}} \left(-E_{k1}\alpha_k \alpha_k^\dagger + E_{k2}\beta_k^\dagger \beta_k \right) + \sum_{\vec{k}} a_k + \frac{\Delta^2}{V}\,; \tag{2.44}$$

In region D_2, we have some depairing and the Hamiltonian is better written as:

$$\mathscr{H}' = \sum_{\vec{k}} \left(E_{k1}\alpha_k^\dagger \alpha_k - E_{k2}\beta_k \beta_k^\dagger \right) + \sum_{\vec{k}} b_k + \frac{\Delta^2}{V}\,. \tag{2.45}$$

Then the ground state will be described by:

$$E_{G,trial} = \sum_{D,\vec{k}} (b_k - E_{k2}) + \sum_{D_1,\vec{k}} a_k + \sum_{D_2\vec{k}} b_k + \frac{\Delta^2}{V}\,, \tag{2.46}$$

which can be reduced to Sarma's solution III in the absence of deformations of the Fermi surfaces. And for the calculation done later, we write the sum and integral over region $D_{1\alpha}$ and $D_{2\beta}$ as D_1 and D_2 for simplicity.

CHAPTER III

TRIAL WAVE FUNCTION AND VARIATIONAL METHOD

A. The Trial Wave Function and the Shift of the Chemical Potential

From the results of the depairing regions, we obtained the following ground state wave function:

$$|\Psi\rangle = \prod_{D,\vec{k}}(u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) \prod_{D_1,\vec{k}} c^\dagger_{k\uparrow} \prod_{D_2,\vec{k}} c^\dagger_{-k\downarrow}|vac\rangle \,, \tag{3.1}$$

where D_1 and D_2 are the two depairing regions, and D, the paired region. This wave function will be used to evaluate the expectation value of the original Hamiltonian for minimization:

$$\begin{aligned} \mathscr{H} \;\; &= \;\; \sum_{\vec{k}}[\frac{\hbar^2 k^2}{2m} - \mu_1']c^\dagger_{k\uparrow}c_{k\uparrow} \\ &\quad + \sum_{-\vec{k}}[\frac{\hbar^2 k^2}{2m} - \mu_2']c^\dagger_{-k\downarrow}c_{-k\downarrow} - \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}c_{-k\downarrow}c_{k\uparrow} \,, \end{aligned} \tag{3.2}$$

where μ_1' and μ_1' are Fermi energies of the electrons of spin-up and -down with Fermi momenta $\vec{k}_1'$ and $\vec{k}_2'$ respectively. And we define:

$$\begin{aligned} \delta\mu' \equiv h \equiv \frac{\mu_1' - \mu_2'}{2} \,, \\ \mu' \equiv \frac{\mu_1' + \mu_2'}{2} \,, \\ \mu_1' \equiv \frac{\hbar^2 k_1'^2}{2m} \,, \\ \mu_2' \equiv \frac{\hbar^2 k_2'^2}{2m} \,, \end{aligned} \tag{3.3}$$

$$\mu \equiv \frac{\hbar^2 k_F'^2}{2m} \,. \tag{3.4}$$

The trial wave function represents a new state with two deformed Fermi surfaces. During the process of deformation, we should have the total number of electrons to be a constant, i.e. $\langle N\rangle_{SC} = \langle N\rangle_{Normal}$, where SC stands for the new superconducting state, and the subscript "Normal", the normal state.

By using the number operator, we obtain:

$$\begin{aligned}
\langle N\rangle_{SC} &= \langle\Psi|c^\dagger_{k\uparrow}c_{k\uparrow} + c^\dagger_{-k\downarrow}c_{-k\downarrow}|\Psi\rangle && (3.5)\\
&= \sum_{D,\vec{k}} 2v_k^2 + \sum_{D_1,\vec{k}} 1 + \sum_{D_2,\vec{k}} 1\,, && (3.6)
\end{aligned}$$

while the particle number for the related normal state is :

$$\langle N\rangle_{Normal} = \sum_{\vec{k}\uparrow} \theta(k_1' - k) + \sum_{-\vec{k}\downarrow} \theta(k_2' - k)\,. \tag{3.7}$$

Since the particles far below the Fermi surface will not be affected when this metal has a phase transition, we can use the weak coupling approximation when calculating the difference between $\langle N\rangle_{SC}$ and $\langle N\rangle_N$, in other words, the particles which are located far below the Fermi surface will be canceled by this method. Then we transform the sum to an integral under the weak coupling approximation as following (for a three dimensional isotropic system):

$$\begin{aligned}
\sum_{\vec{k}} &= \frac{V k_F m}{(2\pi\hbar)^2} \int \mathrm{d}\frac{\hbar^2 k^2}{2m} \int \mathrm{d}\cos\theta \\
&= \frac{N(0)}{2} \int \mathrm{d}\epsilon_k \int \mathrm{d}\cos\theta\,.
\end{aligned} \tag{3.8}$$

Using this approximation in $\langle N\rangle_{SC} - \langle N\rangle_{Normal}$ and integrating the energy over the range $[\mu - \hbar\omega_D, \mu + \hbar\omega_D]$, we obtain:

$$\begin{aligned}
& \langle N\rangle_{SC} - \langle N\rangle_{Normal} \\
= \; & \frac{N(0)}{2}\int_D \mathrm{d}\epsilon_k \int_D \mathrm{d}\cos\theta \quad 2v_k^2 \\
& + \frac{N(0)}{2}\int_{D_1} \mathrm{d}\epsilon_k \int_{D_1} \mathrm{d}\cos\theta + \frac{N(0)}{2}\int_{D_2} \mathrm{d}\epsilon_k \int_{D_2} \mathrm{d}\cos\theta \\
& - \frac{N(0)}{2}(4\mu' - 4\mu + 4\hbar\omega_D)\,.
\end{aligned}$$

The first term in this equation seems a little complicated, but after a few steps, we find it will give terms which will cancel the 2nd and 3rd terms above:

$$\begin{aligned}
1st\;term \; = \; & \frac{Vk_F m}{(2\pi\hbar)^2}[4\hbar\omega_D - \frac{2}{3}(\mu_1\epsilon_1 + \mu_2\epsilon_2)] \\
& - \frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2 k^2}{2m}\int_{D_1} \mathrm{d}\cos\theta - \frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2 k^2}{2m}\int_{D_2} \mathrm{d}\cos\theta\,.
\end{aligned} \tag{3.9}$$

Thus we obtain:

$$\langle N\rangle_s - \langle N\rangle_n = \frac{Vm}{(2\pi\hbar)^2}[8\mu\Delta k_F + 4\hbar\omega_D\Delta k_F - \frac{2}{3}(\mu_1\epsilon_1 + \mu_2\epsilon_2)k_F]\,,$$

where

$$\Delta k_F \equiv k_F - k'_F\,, \tag{3.10}$$

$$\Delta\mu \equiv \mu - \mu'\,. \tag{3.11}$$

In order to keep the number of particles fixed, we require $\langle N\rangle_{SC} - \langle N\rangle_{Normal} = 0$, so we have:

$$8\mu\Delta k_F + 4\hbar\omega_D\Delta k_F - \frac{2}{3}(\mu_1\epsilon_1 + \mu_2\epsilon_2)k_F = 0\,,$$

which gives

$$\frac{\Delta k_F}{k_F} \simeq \frac{1}{12}\frac{(\mu_1\epsilon_1 + \mu_2\epsilon_2)}{\mu}\,. \tag{3.12}$$

So the shift of chemical potential is proportional to $\mu_1\epsilon_1 + \mu_2\epsilon_2$. If we have $\mu_1\epsilon_1 + \mu_2\epsilon_2 = 0$, both the chemical potential and total number of electrons will be fixed.

B. The Ground State Energy and Minimization

In the following derivation, $\delta\mu$, μ, $\frac{\mu_1\epsilon_1+\mu_2\epsilon_2}{2}$, and $\frac{\mu_1\epsilon_1-\mu_2\epsilon_2}{2}$ will be our variables. We will always keep $\delta\mu \geq 0$, $\epsilon_1 \geq 0$ and $\epsilon_2 \leq 0$ to assure $\mu_1 \geq \mu_2$ and the deformations of the two Fermi surfaces are opposite in "direction". From Sarma's conclusion, we can see that the superconducting state never prefers $h > \Delta_{BCS}$. So we'd better confine z, which is equivalent to $\frac{\mu_1\epsilon_1-\mu_2\epsilon_2}{2}$, to a reasonable value in unit of Δ_{BCS}. And also noting that the Fermi surface is much larger than $\hbar\omega$ while $\hbar\omega$ is much larger than Δ_{BCS}. For example, for most metals like V, Zn, Nb and etc., their gap Δ are around 1 meV, their $\hbar\omega_D$ are around 25 meV (in terms of the Debye temperature it is around 300K) and the Fermi energy is around 1 eV. So if we use ϵ_1, ϵ_2, μ_1 and μ_2 instead of z and $\delta\mu$, we will find ϵ_1 and ϵ_2 should be a very small value around 0.001. There is one more reason if we take a look at the trial gap equation given by Eq. (2.6):

$$ln\frac{\Delta_{BCS}}{\Delta} = \frac{1}{2}\int\limits_{D_1+D_2} \mathrm{d}\cos\theta \; ln\frac{\sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2} + |\delta\mu - z\cos^2\theta|}{\Delta}\,. \tag{3.13}$$

Comparing to the gap equation Eq. (1.22) of the FF state, we will find z and $\delta\mu$ playing a very similar role as Q and h. Also based on the relative values of the gap, the Debye temperature, and the Fermi energy, any energy value, in the unit of Δ_{BCS}, around $\Delta_{BCS}/\hbar\omega_D$ or smaller will be discarded.

Now we can use the trial wave function to calculate the expectation value of the original Hamiltonian. After a complicated calculation, we obtain:

$$\begin{aligned} & \langle\Psi|\mathscr{H}|\Psi\rangle \\ = & \sum_{D,\vec{k}}(\frac{\hbar^2k^2}{2m} - \mu_1')|v_k|^2 + \sum_{D_1,\vec{k}}(\frac{\hbar^2k^2}{2m} - \mu_1') \\ & + \sum_{D,\vec{k}}(\frac{\hbar^2k^2}{2m} - \mu_2')|v_k|^2 + \sum_{D_2,\vec{k}}(\frac{\hbar^2k^2}{2m} - \mu_2') - \sum_{D,\vec{k},\vec{k}'} V_{k,k'}u_k v_{k'} v_k u_{k'}\,. \end{aligned}$$

For an S-wave superconductor, the coupling constant is momentum independent. Then we obtain:

$$\begin{aligned} & \langle\Psi|\mathscr{H}|\Psi\rangle_s \\ = & \sum_{\vec{k}} 2\epsilon_k'|v_k|^2 + \sum_{D_1+D_2,\vec{k}} \epsilon_k'(1-2|v_k|^2) - \sum_{D_1,\vec{k}} h + \sum_{D_2,\vec{k}} h - V\left(\sum_{D,\vec{k}} u_k v_k\right)^2\,, \end{aligned}$$

where

$$\epsilon_k' = \epsilon_k + \Delta\mu = \frac{\hbar^2k^2}{2m} - \mu_1'\,. \tag{3.14}$$

In the above equation we can see that the first term is the energy due to the quasiparticles, the second term ,due to depairng, which always increases the energy, the 3rd and 4th terms are the Zeeman energy, and the last one, which is negative, is the interaction-energy contribution. We also can find some reasonable symmetries: $-h \leftrightarrow h$, $D_1 \leftrightarrow D_2$.

Then we replace summations by integrals, and plug in the expression for $|v_k|^2$ and integrate energy over the region $[\mu - \hbar\omega_D, \mu + \hbar\omega_D]$. The expectation value, which will be denoted as $E_{SC}(0)$, will become

$$
\begin{aligned}
& E_{SC}(0) \\
= \; & N(0)\Bigg(\Delta^2 ln\frac{2\hbar\omega_D}{\Delta} - \hbar^2\omega_D^2 + \frac{1}{5}(\mu_1\epsilon_1 + \mu_2\epsilon_2)^2 - \frac{\Delta^2}{2} \\
& +2\Delta\mu\hbar\omega_D - \frac{2}{3}\Delta\mu(\mu_1\epsilon_1 + \mu_2\epsilon_2)\Bigg) \\
& +\frac{N(0)}{2}\int\limits_{D_1+D_2}\Bigg(- \Delta^2 ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta}) \\
& +|\delta\mu - z\cos^2\theta|\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\Bigg)\,\mathrm{d}x \\
& -\frac{N(0)}{2}\int\limits_{D_1} 2h\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x \\
& +\frac{N(0)}{2}\int\limits_{D_2} 2h\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x \\
& -VN(0)^2\Delta^2\Bigg(ln\frac{2\hbar\omega_D}{\Delta} \\
& -\frac{1}{2}\int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Bigg)^2,
\end{aligned}
\tag{3.15}
$$

where $x \equiv \cos\theta$. Its difference with the normal-state energy $E_{Normal}(0)$ is given by:

$$
\begin{aligned}
& E_{SC}(0) - E_{Normal}(0) \\
= \; & N(0)\left(\Delta^2 ln\frac{2\hbar\omega_D}{\Delta} + \frac{7}{60}(\mu_1\epsilon_1 + \mu_2\epsilon_2)^2 - \frac{\Delta^2}{2}\right) \\
& + \frac{N(0)}{2}\int\limits_{D_1+D_2}\Bigg(-\Delta^2 ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx2\,)^2 - \Delta^2}}{\Delta}) \\
& + |\delta\mu - zx^2|\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\Bigg)\,\mathrm{d}x \\
& - \frac{N(0)}{2}\int\limits_{D_1} 2h\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x + \frac{N(0)}{2}\int\limits_{D_2} 2h\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x \\
& - V\Bigg(N(0)\Delta ln\frac{2\hbar\omega_D}{\Delta} \\
& - \frac{N(0)}{2}\int\limits_{D_1+D_2}\Delta ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Bigg)^2 .
\end{aligned}
\tag{3.16}
$$

From the above equation, we can see the energy difference prefers $\mu_1\epsilon_1 + \mu_2\epsilon_2 = 0$, which means if the total number of particle is fixed, the ground state energy will prefer a unchanged chemical potential. We obtain the gap equation, which is the 1st derivative of $E_{SC}(0) - E_{Normal}(0)$ with respect to Δ, and also the 2nd derivative, as well as the first derivative of $E_{SC}(0) - E_{Normal}$ with respect to $\delta\mu$ and z:

$$
\begin{aligned}
& \frac{\partial(E_{SC}(0) - E_{Normal}(0))}{\partial \Delta} \\
= \; & 2N(0)\Delta(ln\frac{2\hbar\omega_D}{\Delta} - 1) \\
& -N(0)\Delta \int\limits_{D_1+D_2} ln\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta}\,\mathrm{d}x \\
& +N(0)\Delta\int\limits_{D_1} \frac{h}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x - N(0)\Delta\int\limits_{D_2} \frac{h}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x \\
& -2VN(0)^2\Delta\Bigg(ln\frac{2\hbar\omega_D}{\Delta} \\
& -\frac{1}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\Bigg) \\
& \times\Bigg(ln\frac{2\hbar\omega_D}{\Delta} - 1 - \frac{1}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - z\cos^2\theta| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x \\
& +\frac{1}{2}\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x\Bigg) = 0\,,
\end{aligned}
\tag{3.17}
$$

which is our gap equation.

And

$$
\begin{aligned}
&\frac{\partial(E_{SC}(0)-E_{Normal}(0))}{\partial\delta\mu}\\
= \;& N(0)\int\limits_{D_1}\sqrt{(\delta\mu-zx^2)^2-\Delta^2}\,\mathrm{d}x - N(0)\int\limits_{D_2}\sqrt{(\delta\mu-zx^2)^2-\Delta^2}\,\mathrm{d}x\\
&-N(0)\int\limits_{D_1+D_2}\frac{h|\delta\mu-zx^2|}{\sqrt{(\delta\mu-zx^2\,)^2-\Delta^2}}\,\mathrm{d}x\\
&+N(0)\Delta^2\Bigg(VN(0)ln\frac{2\hbar\omega_D}{\Delta}\\
&-\frac{VN(0)}{2}\int\limits_{D_1+D_2}ln\big(\frac{|\delta\mu-zx^2|+\sqrt{(\delta\mu-zx^2\,)^2-\Delta^2}}{\Delta}\big)\,\mathrm{d}x\Bigg)\\
&\times\Bigg(\int\limits_{D_1}\frac{1}{\sqrt{(\delta\mu-zx^2\,)^2-\Delta^2}}\,\mathrm{d}x-\int\limits_{D_2}\frac{1}{\sqrt{(\delta\mu-zx^2\,)^2-\Delta^2}}\,\mathrm{d}x\Bigg)=0\,,
\end{aligned}
\tag{3.18}
$$

And

$$
\begin{aligned}
&\frac{\partial(E_{SC}(0)-E_{Normal}(0))}{\partial z}\\
= \;& -N(0)\int\limits_{D_1}\sqrt{(\delta\mu-zx^2)^2-\Delta^2}x^2\,\mathrm{d}x + N(0)\int\limits_{D_2}\sqrt{(\delta\mu-zx^2)^2-\Delta^2}x^2\,\mathrm{d}x\\
&+N(0)\int\limits_{D_1+D_2}\frac{h|\delta\mu-zx^2|x^2}{\sqrt{(\delta\mu-zx^2\,)^2-\Delta^2}}\,\mathrm{d}x+N(0)\Delta^2\Bigg(VN(0)ln\frac{2\hbar\omega_D}{\Delta}\\
&-\frac{VN(0)}{2}\int\limits_{D_1+D_2}ln\big(\frac{|\delta\mu-zx^2|+\sqrt{(\delta\mu-zx^2\,)^2-\Delta^2}}{\Delta}\big)\,\mathrm{d}x\Bigg)\\
&\times\Bigg(-\int\limits_{D_1}\frac{x^2}{\sqrt{(\delta\mu-zx^2\,)^2-\Delta^2}}\,\mathrm{d}x+\int\limits_{D_2}\frac{x^2}{\sqrt{(\delta\mu-zx^2\,)^2-\Delta^2}}\,\mathrm{d}x\Bigg)=0\,,
\end{aligned}
\tag{3.19}
$$

And

$$
\begin{aligned}
&\frac{\partial^2(E_{SC}(0)-E_{Normal}(0))}{\partial^2\Delta}\\
= &\ 2N(0)(ln\frac{2\hbar\omega_D}{\Delta}-2)\\
&-N(0)\int\limits_{D_1+D_2} ln\frac{|\delta\mu-zx^2|+\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}}{\Delta}\,\mathrm{d}x\\
&+N(0)\int\limits_{D_1+D_2}\frac{|\delta\mu/z-x^2|}{\sqrt{(\delta\mu/z-x^2\ )^2-\Delta^2/z^2}}\,\mathrm{d}x\\
&+N(0)\int\limits_{D_1}\frac{h}{\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}}\,\mathrm{d}x-N(0)\int\limits_{D_2}\frac{h}{\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}}\,\mathrm{d}x\\
&+N(0)\int\limits_{D_1}\frac{h\Delta^2}{\left(\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}\right)^3}\,\mathrm{d}x-N(0)\int\limits_{D_2}\frac{h\Delta^2}{\left(\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}\right)^3}\,\mathrm{d}x\\
&-2VN(0)^2\Bigg(ln\frac{2\hbar\omega_D}{\Delta}-1+\frac{1}{2}\int\limits_{D_1+D_2}\frac{|\delta\mu-zx^2|}{\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}}\,\mathrm{d}x\\
&-\frac{1}{2}\int\limits_{D_1+D_2} ln(\frac{|\delta\mu-zx^2|+\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}}{\Delta})\,\mathrm{d}x\Bigg)^2\\
&-2VN(0)^2\Bigg(-1+\frac{1}{2}\int\limits_{D_1+D_2}\frac{|\delta\mu-zx^2|^3}{\left(\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}\right)^3}\,\mathrm{d}x\Bigg)\\
&\times\Bigg(ln\frac{2\hbar\omega_D}{\Delta}-\frac{1}{2}\int\limits_{D_1+D_2} ln(\frac{|\delta\mu-zx^2|+\sqrt{(\delta\mu-zx^2\ )^2-\Delta^2}}{\Delta})\,\mathrm{d}x\Bigg)\geq 0\,.
\end{aligned}
\tag{3.20}
$$

A second look at $E_(SC)(0)-E_{Normal}(0)$ and the gap equation will tell us there are at least 4 cases:

Case 1: When $\Delta>\delta\mu$ and $\frac{\Delta+\delta\mu}{z}>1$, there will be no depairing region and we will get the unpolarized BCS ground state and the BCS gap. Especially when $z\leq 1$

and $\delta\mu \leq 1$, the BCS ground state is always one of the solutions.

Case 2: When $\Delta > \delta\mu$ and $\frac{\Delta+\delta\mu}{z} < 1$, there is only the depairng region D_2. In this case the Fermi energy difference z is at least twice larger than $\delta\mu$. No such region is found when z and $\delta\mu$ are below unity.

Case 3: When $\Delta < \delta\mu$ and $\frac{\Delta+\delta\mu}{z} > 1$, there is only the depairing region D_1. Sarma's solution III is just a special case when $z = 0$ and $\delta\mu = h$.

Case 4: When $\Delta < \delta\mu$ and $\frac{\Delta+\delta\mu}{z} < 1$, those two depairng regions will both exist.

I have written a C program, which is attached at the end as an appendix, to investigate the gap Δ over a lattice of $\delta\mu$ and z values by using the gap equation. We select Δ_{BCS} to be the unit and a typical value $N(0) = 0.2$ for weak coupling approximation. From the data, we find that the minimum energy exists when $z \leq 1$ and $\delta\mu \leq 1$. Case 2 never exists in this region and Case 4 only exists when the gap Δ become very small. For large z and $\delta\mu$ above 1 but less than 3, the energy $E_{SC}(0) - E_{Normal}(0)$ will be larger than 0, which means no superconducting state exists within this region. Figures 5 to 15 are plotted based on the results.

These figures show for energy that the unpolarized BCS state is always the lowest energy state when h is below the Clogston-Chandrasekhar limit, which is $\Delta_{BCS}/\sqrt{2}$. We didn't show the picture for h below $0.5\Delta BCS$, but we reach the same conclusion from this region of h. For the other case when h is below the Clogston-Chandrasekhar limit but above $h = 0.5\Delta BCS$, the system will prefer $\delta\mu = h$ and $z = 0$, which means no deformation at all. When it is above the Clogston-Chandrasekhar limit, we still have local minimum at $\delta\mu = h$ and $z = 0$, but it is no longer the global minimum. That's because the depairing region D_2 plays a more and more important role as

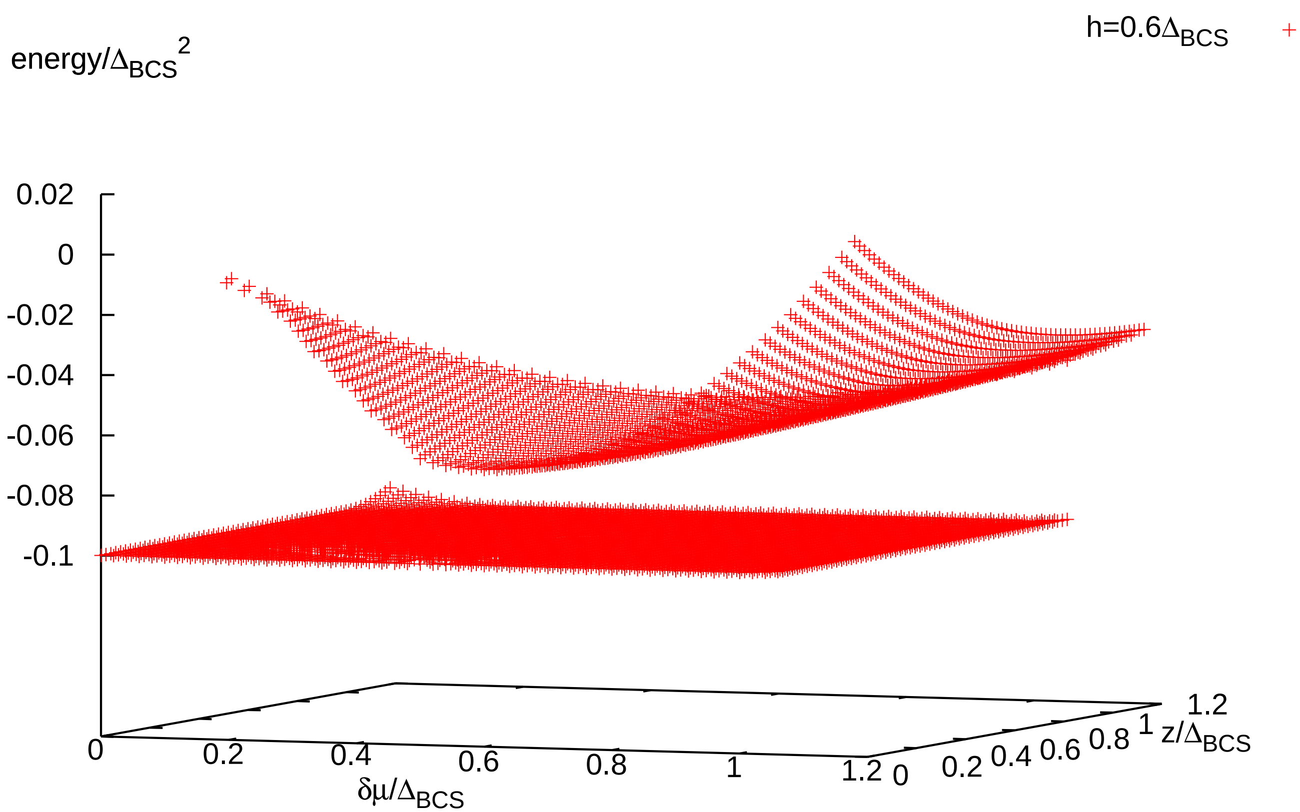

Fig. 5. 3D plot of the energy of the DFS state when the external magnetic field h=0.6Δ_{BCS} and $N(0)V$=0.2.

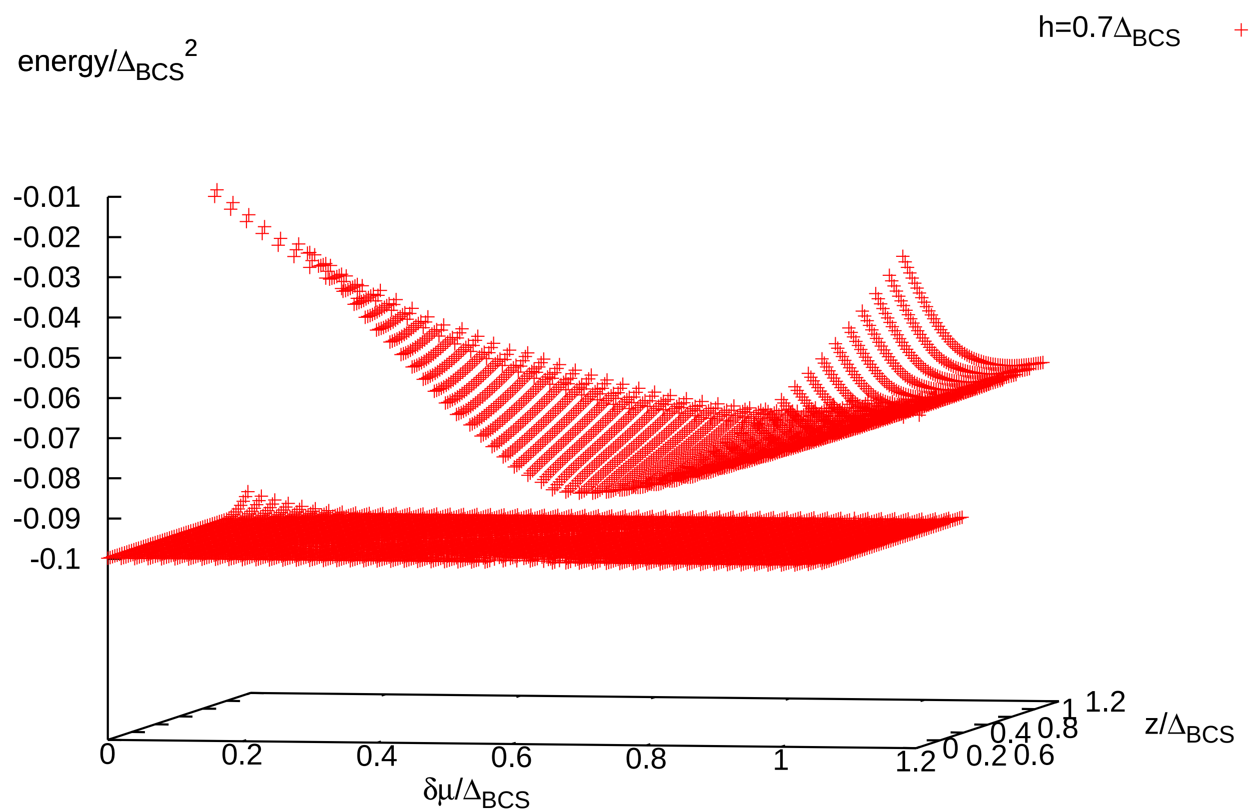

Fig. 6. 3D plot of the energy of the DFS state when the external magnetic field h=0.7Δ_{BCS} and $N(0)V$=0.2.

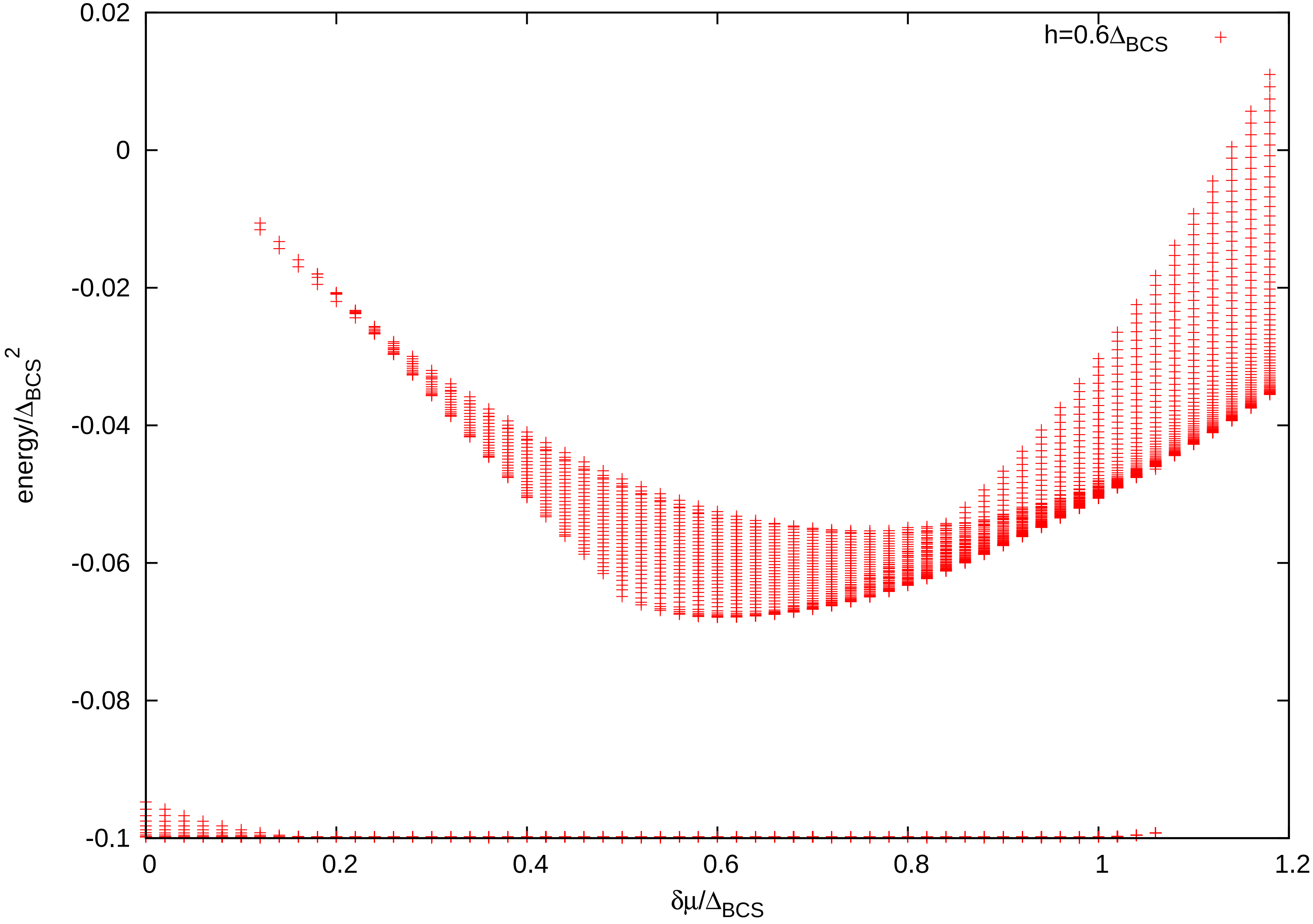

Fig. 7. 2D plot of the energy of the DFS state when the external magnetic field h=0.6Δ_{BCS} and $N(0)V = 0.2$. We can see the minimum is at $\Delta\mu$=0.6.

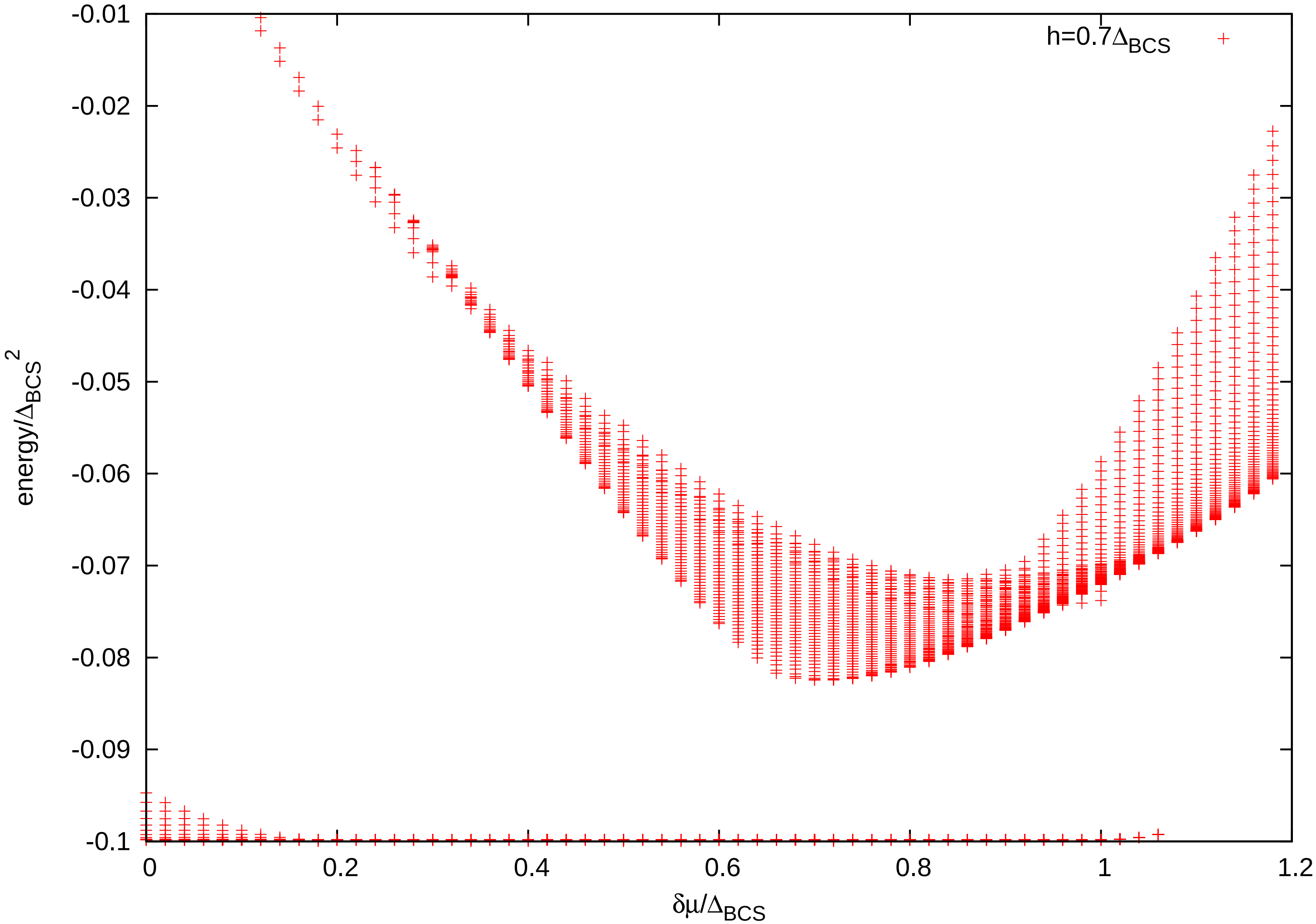

Fig. 8. 2D plot of the energy of the DFS state when the external magnetic field h=0.7Δ_{BCS} and $N(0)V = 0.2$. We can see that the minimum is at $\Delta\mu$=0.7.

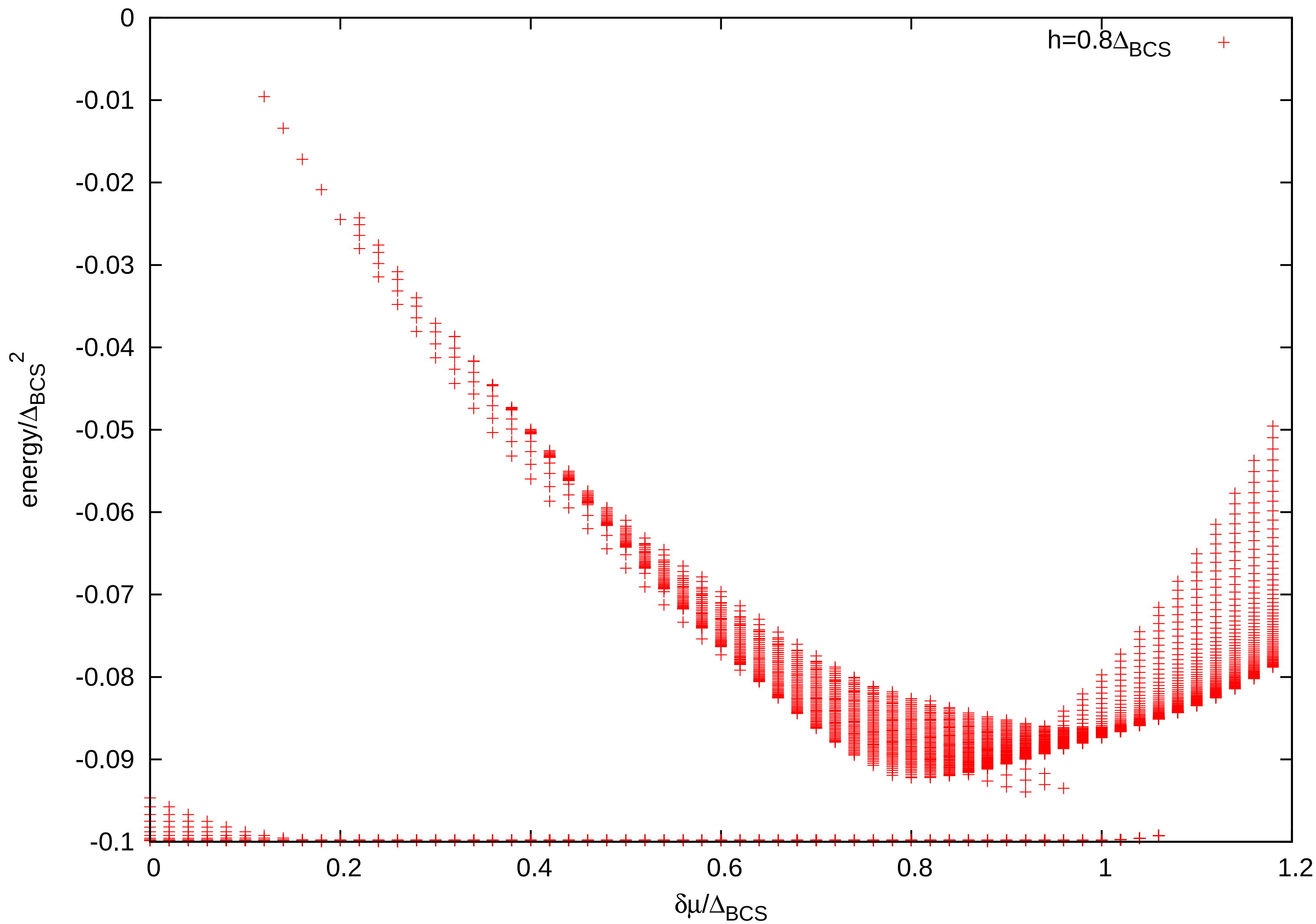

Fig. 9. 2D plot of the energy of the DFS state when the external magnetic field h=0.8Δ_{BCS} and $N(0)V = 0.2$. We can see a local minimum at $\Delta\mu$=0.8, but it is no longer the minimum energy for $\Delta < 1$.

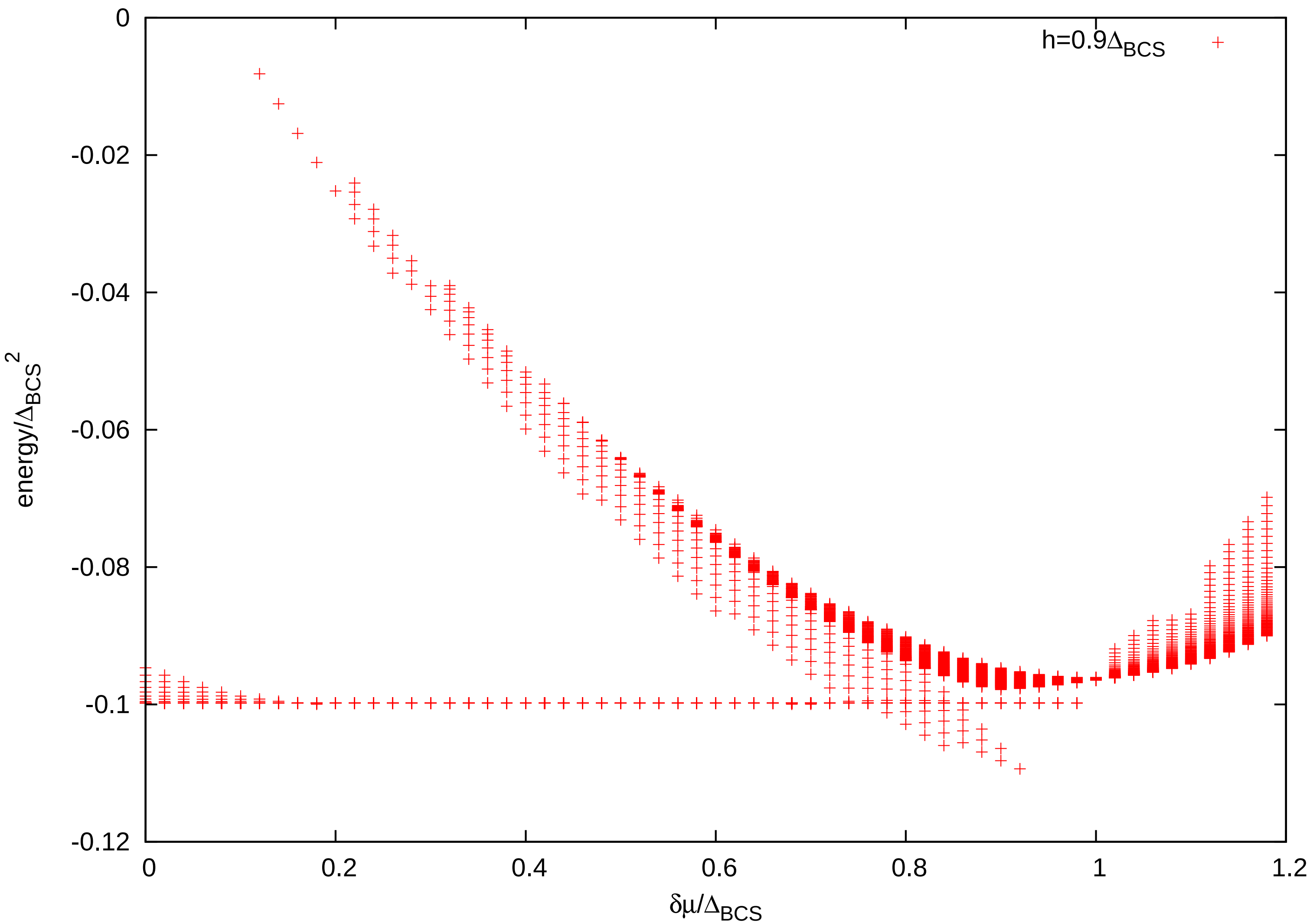

Fig. 10. 2D plot of the energy of the DFS state when the external magnetic field h=0.9Δ_{BCS} and $N(0)V = 0.2$. We can see a local minimum at $\Delta\mu$=0.9, but it is no longer the minimum energy for $\Delta < 1$.

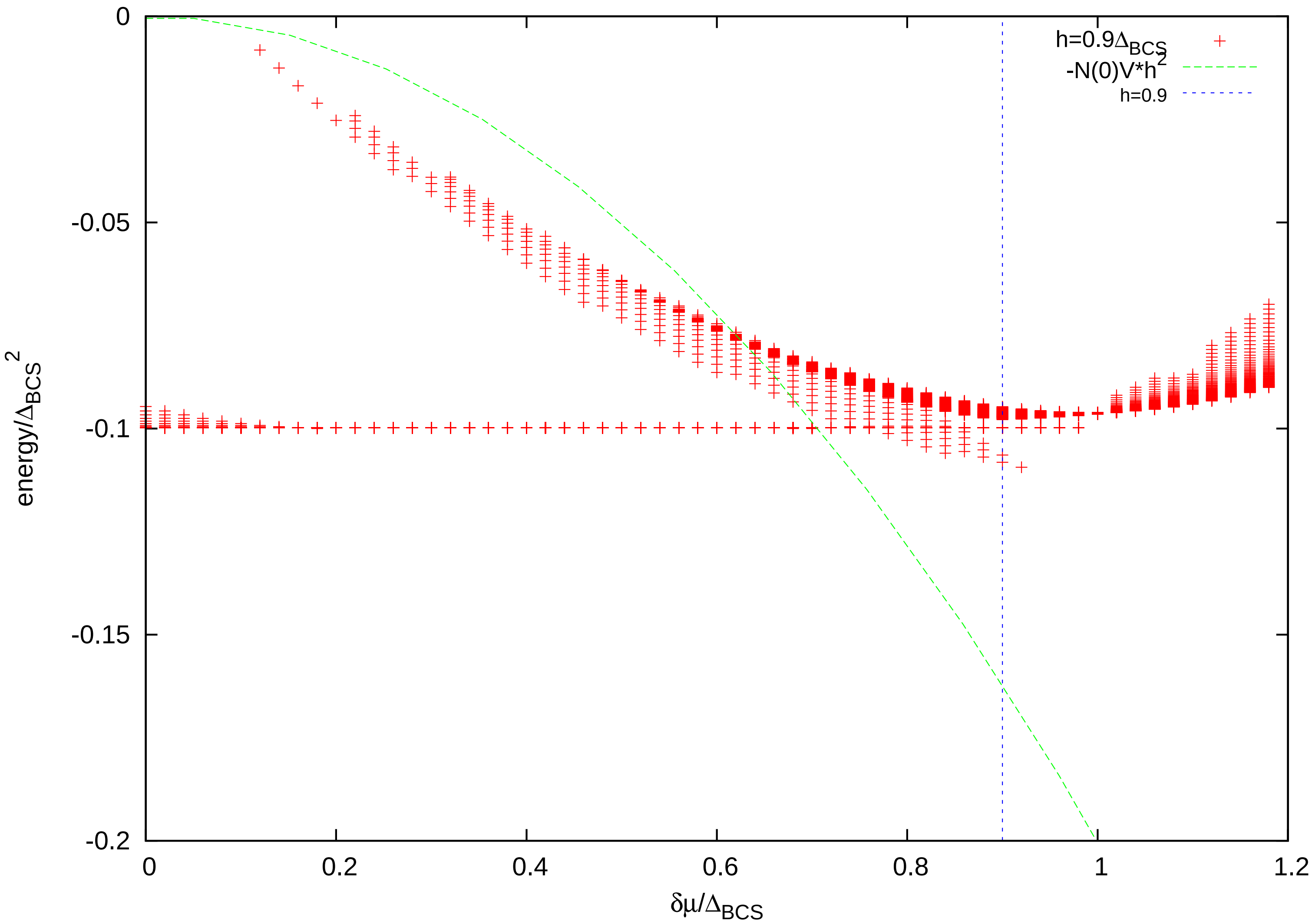

Fig. 11. Identical plot for Fig. 10. We can see that the energy of the normal state with Pauli paramagnetism is the lowest one.

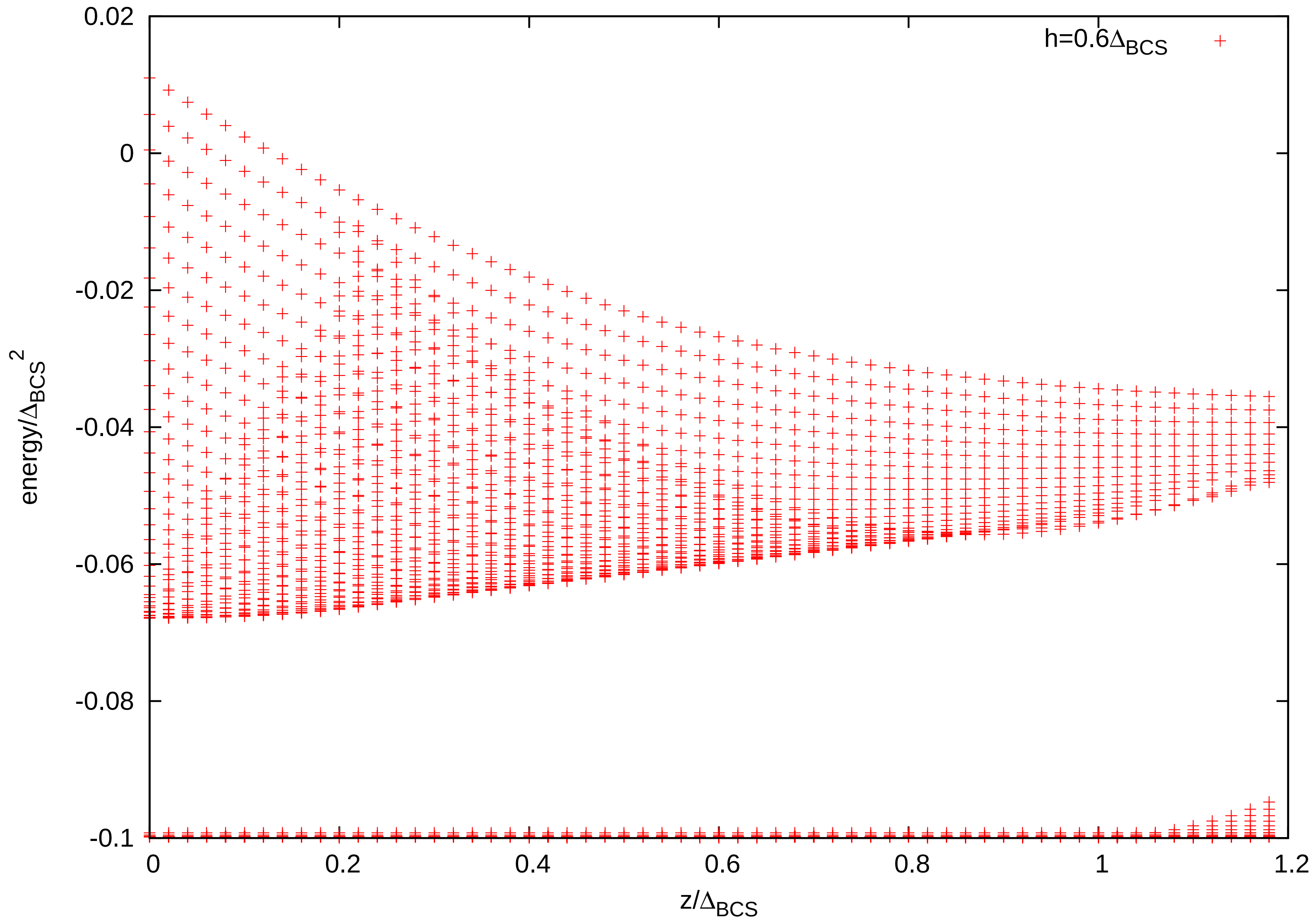

Fig. 12. 2D plot of the energy of the DFS state when the external magnetic field h=0.6Δ_{BCS} and $N(0)V = 0.2$. We can see that the minimum is at z=0.

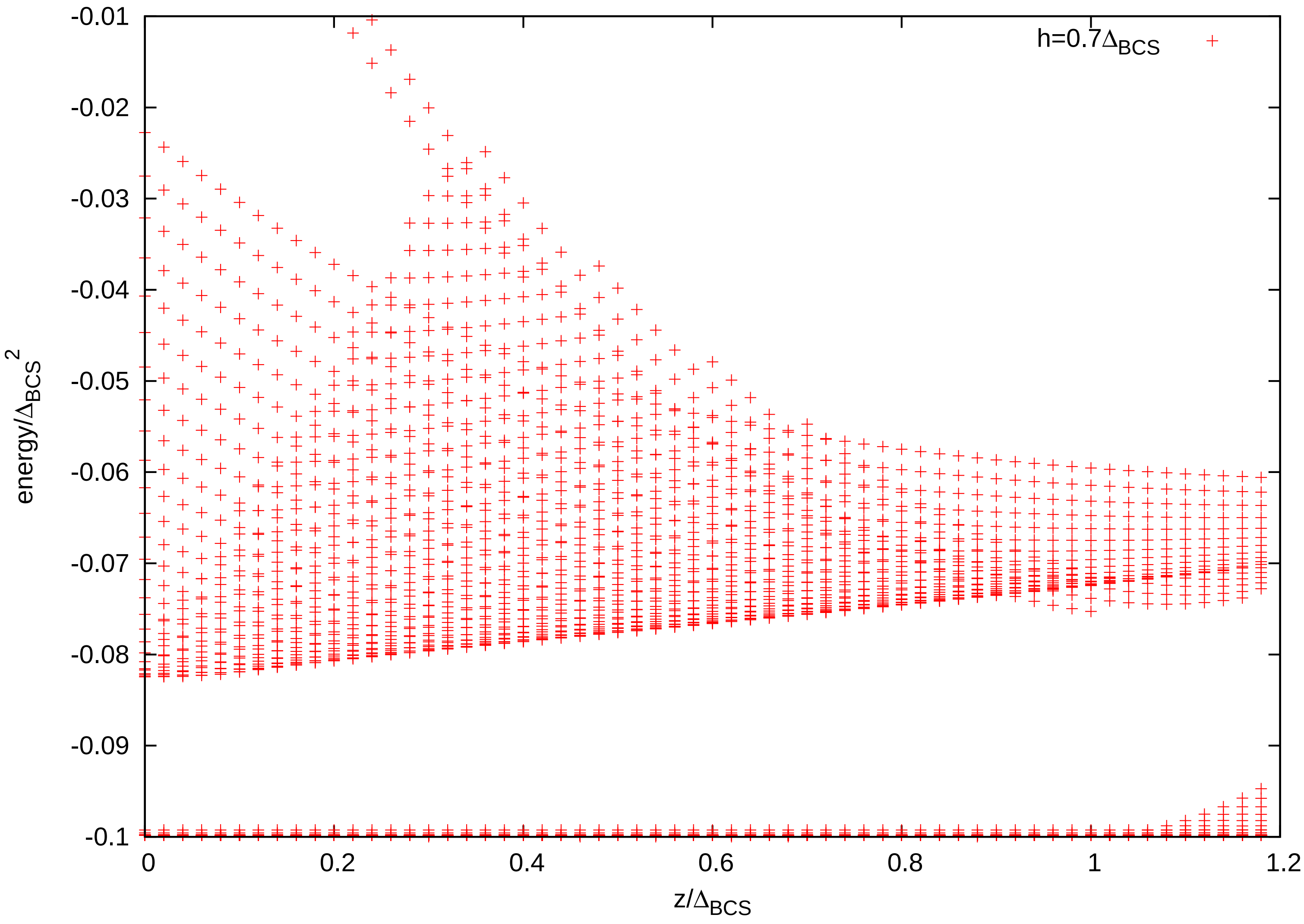

Fig. 13. 2D plot of the energy of the DFS state when the external magnetic field h=0.7Δ_{BCS} and $N(0)V = 0.2$. We can see that the minimum is at z=0.

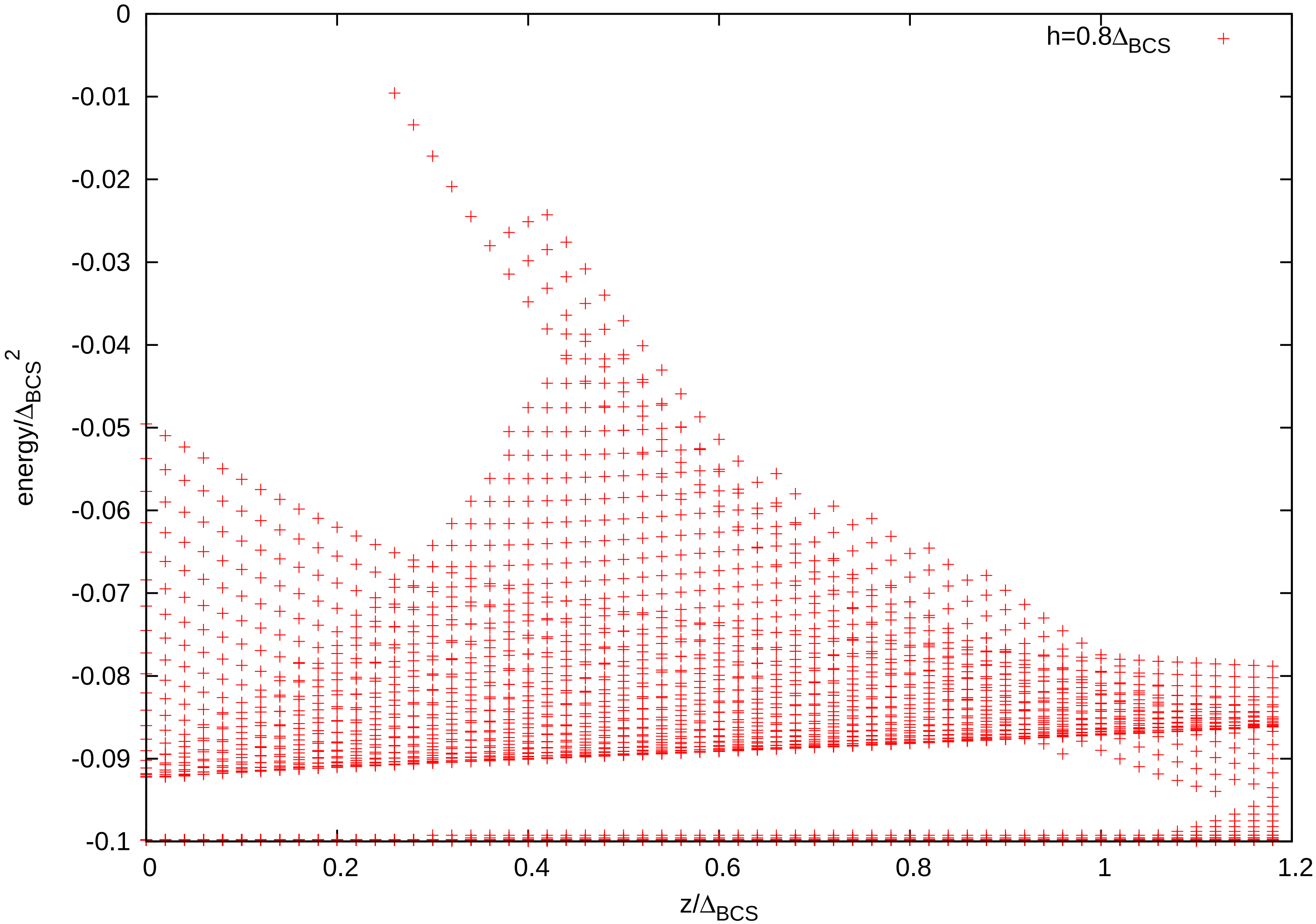

Fig. 14. 2D plot of the energy of the DFS state when the external magnetic field h=0.8Δ_{BCS} and $N(0)V = 0.2$. We can see that a local minimum is at z=0, but it is not the global minimum.

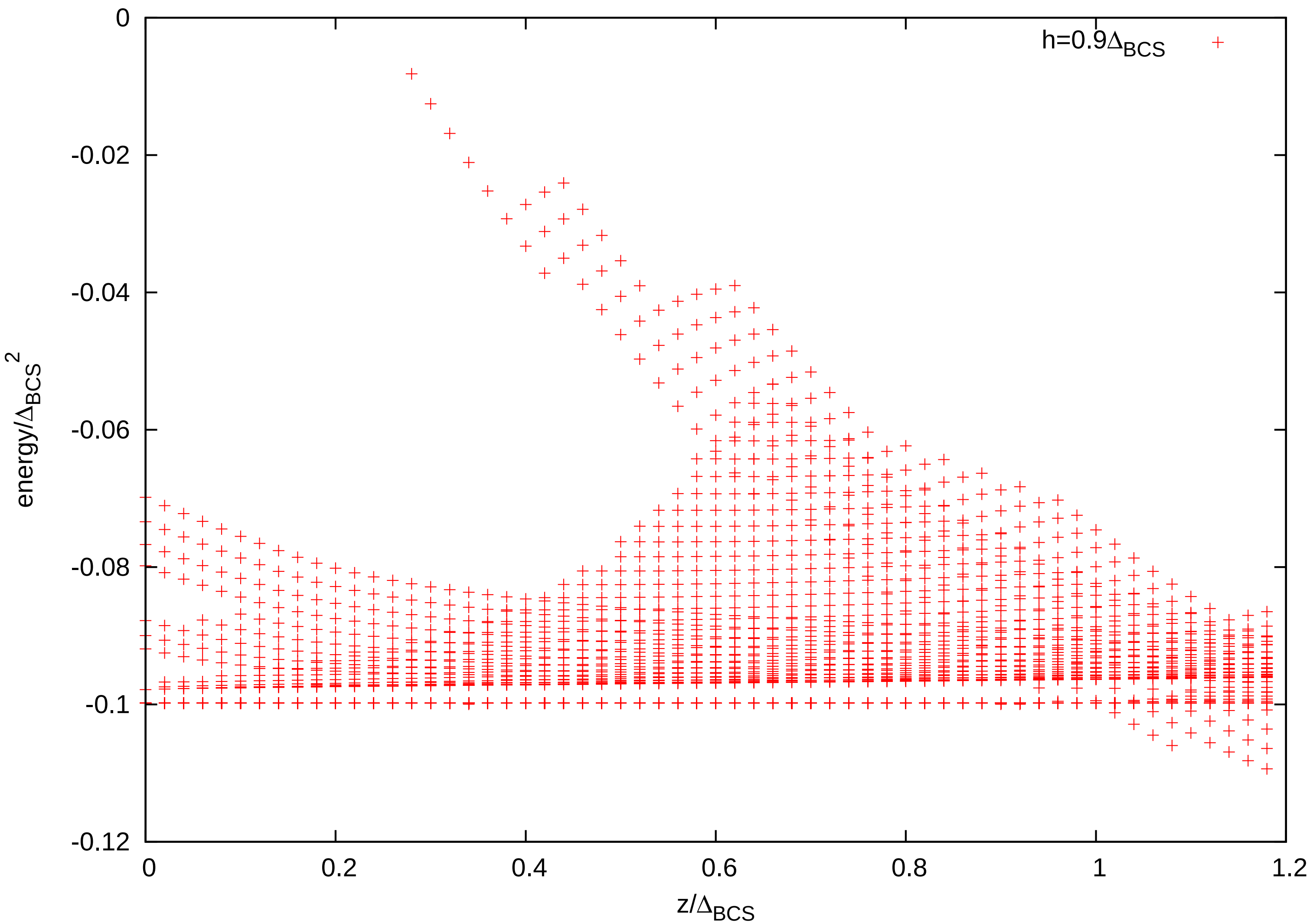

Fig. 15. 2D plot of the energy of the DFS state when the external magnetic field h=0.9Δ_{BCS} and $N(0)V = 0.2$. There is a very shallow local minimum is at z=0, but it is not the global minimum.

h increases. The appearance of region D_2 decreases the gap Δ and increases the Zeeman term. At this time, the region D_1 is still larger than region D_2, so the energy is lowered. But if the region D_2 keeps on increasing, it will lead to a smaller region D_1, or, in other words, more and more electrons will move from the original larger Fermi surface to the smaller one. Both from physics and Eq. (3.16), we can see that the energy will increase.

CHAPTER IV

CONCLUSION

We investigated, at zero temperature only, the deformed-Fermi-surfaces (DFS) mechanism for pairing of fermions with mismatched Fermi surfaces — an idea first proposed by M*ü*ther and Sedrakian, by using a variational method where the ground-state wave-function is obtained by diagonalizing the following "trial Hamiltonian":

$$\begin{aligned}\mathscr{H}_{trial} &= \sum_{\vec{k}\uparrow}[\frac{\hbar^2 k^2}{2m} - \mu_1(1-\epsilon_1\cos^2\theta)]c^\dagger_{k\uparrow}c_{k\uparrow} \\ &+ \sum_{-\vec{k}\downarrow}[\frac{\hbar^2 k^2}{2m} - \mu_2(1-\epsilon_2\cos^2\theta)]c^\dagger_{-k\downarrow}c_{-k\downarrow} - \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}c_{-k\downarrow}c_{k\uparrow}\end{aligned} \quad (4.1)$$

The depairing regions have to be carefully analyzed in the whole parameter space, before the energy gap for elementary excitations and the total energy of the system can be studied in the whole parameter space. Due to the relative values of the Fermi energy, Debye energy and energy gap of most low temperature superconductors, we find that $\delta\mu \equiv (\mu_1 - \mu_2)/2$ and $z \equiv (\mu_1\epsilon_1 - \mu_2\epsilon_2)/2$ are better variables to characterize the variational state. Within the regions defined by $0 \leq \delta\mu \leq 1$ and $0 \leq z \leq 1$, we find that the energy of the DFS state is always higher than the energy of the BCS Ground state or that of the normal state with Pauli paramagnetism. So the DFS state is not a preferred state under the weak-coupling approximation. However, this conclusion is obtained under the assumption that the system being considered is a conductor subject to a uniform exchange or Zeeman field, so that the externally controlled variables are the electron density n and the Zeeman energy

h, which is just the electron magnetic moment times the external magnetic field. In trapped fermionic atomic mixtures where the two species of atoms are not interchangeable, or in proton-neutron pairing in nuclear physics, or in hetero-quark pairing in particle physics, where the the externally controllable variables might be n_1 and n_2, the number densities of the two species of fermions doing pairing, rather than $n = n_1 + n_2$ and h, the study must be redone, and the conclusion can still be that this idea can still win, as the unpolarized BCS state will no longer be an available option. On the other hand, we have so far considered spatially uniform solutions only, and therefore have not entertained the possibility of any inhomogeneous states including phase separation. This assumption is easily satisfied in the electron system, but not necessarily so in the other systems.

Even for conductors, if the system favors a two dimensional d-wave superconducting state at low temperatures, for example, for which the order parameter is proportional to $\cos 2\theta$, where θ is the angular position of a point on a two dimensional Fermi surface measured relative to the a-axis of a tetragonal crystal structure, a Fermi-surface distortion which is large where the gap is large might still enhance the energy of this state to make it a favorable state, so this proposed state might be preferred state in some circumstances. Thus more study of this model would be warranted.

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APPENDIX A

A. Some Mathematics Used in Chapter III

If we have

$$
\begin{aligned}
F(x) &= \int_{\alpha(x)}^{\beta(x)} f(t,x)\,dt \\
&= E(t,x)\Big|_{\alpha(x)}^{\beta(x)}
\end{aligned}
\tag{A.1}
$$

then

$$
\begin{aligned}
\frac{dF(x)}{dx} &= \frac{dE(t,x)}{dx}\Big|_{\alpha(x)}^{\beta(x)} + \frac{dE(t,x)}{dt}\Big|_{t=\beta(x)}\frac{d\beta(x)}{dx} - \frac{dE(t,x)}{dt}\Big|_{t=\alpha(x)}\frac{d\alpha(x)}{dx} \\
&= \int_{\alpha(x)}^{\beta(x)} \frac{df(t,x)}{dx}\,dt \\
&\quad + \frac{dE(t,x)}{dt}\Big|_{t=\beta(x)}\frac{d\beta(x)}{dx} - \frac{dE(t,x)}{dt}\Big|_{t=\alpha(x)}\frac{d\alpha(x)}{dx} \\
&= \int_{\alpha(x)}^{\beta(x)} \frac{df(t,x)}{dx}\,dt \\
&\quad + f(t,x)\Big|_{t=\beta(x)}\frac{d\beta(x)}{dx} - f(t,x)\Big|_{t=\alpha(x)}\frac{d\alpha(x)}{dx}
\end{aligned}
\tag{A.2}
$$

B. C Program

```
all:newmodified4 newmodified4: newmodified4.cc integrallog.cc
integralinverse.cc integralinverse2.cc integralsqrt.cc
integralsqrt2.cc intsqrtdown.cc intsqrtdown2.cc intsqrtup.cc
inttripleinverse.cc inttripleinverse2.cc
    g++ -o newmodified4 newmodified4.cc integrallog.cc
        integralinverse.cc integralinverse2.cc integralsqrt.cc
        integralsqrt2.cc intsqrtdown.cc intsqrtdown2.cc
        intsqrtup.cc inttripleinverse.cc inttripleinverse2.cc
    g++ -o2 newmodified4 newmodified4.cc integrallog.cc
        integralinverse.cc integralinverse2.cc integralsqrt.cc
        integralsqrt2.cc intsqrtdown.cc intsqrtdown2.cc
        intsqrtup.cc inttripleinverse.cc inttripleinverse2.cc
```

```
#include <iostream>
#include <cmath>
#include <cstdlib>
using namespace std;

double integrallog(double al,double bl,double cl,
                   double dl){
  // a=deltamu/z  and b=delta/z should be given,
  //constant in the function
  // c,d are the boundary of the range, should be given
  // this is the integral of
  //ln(|a-x^2|+\sqrt{(a-x^2)^2-b^2)})-ln(b) over [c,d]
  //and c>0,d<1,
  //these should be determined before this file
  double step=0.00002;
  double hl=step/2;
  double range=dl-cl;
  double number=2*floor(range/step);
  // the range is the divide into 2 parts, the 1st
  // part will be integrated by simpson's rule,the
  //second part will using mid
  //-point rule
  int i,j;
  i=1;
  double x,x1,y,y1,y2,y3;
  if(range<=step) return y=0;
  else
    {
      if(cl<0||dl>1)
      {
       cout<<"boundary exceed"<<endl;
       return 1;
      }
      x=cl;
      x1=x*x;
      y1=abs(al-x1);
      y2=y1*y1-bl*bl;
      y3=sqrt(y2);
      y=log(y1+y3);//initial value
      j=2;
      while (i<=number-1) //step through the iteration
      {
```

```
        j=6-j; //give the 4,2,4,2.....,
        x1=x*x;
        y1=abs(al-x1);
        y2=y1*y1-bl*bl;
        y3=sqrt(y2);
        y=y+j*log(y1+y3);
        i++;
      }
      x=cl+number*hl;
      x1=x*x;
      y1=abs(al-x1);
      y2=y1*y1-bl*bl;
      y3=sqrt(y2);
      y=y+log(y1+y3);
      y=y*hl/3;//end of 1st part
      //2nd part integral
      x=0.5*dl+0.5*cl+0.5*number*hl;
      x1=x*x;
      y1=abs(al-x1);
      y2=y1*y1-bl*bl;
      y3=sqrt(y2);
      y2=log(y1+y3);
      y=y+y2*(dl-cl-number*hl)-range*log(bl);
      //finish ln()-ln(b) numerical integration
      return y;
    }
  }
```

```
#include <iostream>
#include <cmath>
#include <cstdlib>
using namespace std;

double integralinverse(double ai,double bi,
                       double ci,double di) {
  // ai=deltamu/z and bi=delta/z
  // should be given,constant in function
  //(ci,di) is the
  //integral range should be given, and ci>0,di<1,
  //this should be determined before this file
  double stepp=0.00002;
  double hi=stepp/2;
  double rangee=di-ci;
  double numberr=2*floor(rangee/stepp);// the range
   //is then divvided into 2 parts.
  //the 1st part will be integrated by simpson's
  //rule, the second part will
  // using mid-point rule
  // cout<<numberr<<"number inverse"<<endl;
  int i,j;
  i=1;
  double x,x1,y,y1,y2,y3;
  if(rangee<=stepp) return y=0;
  else{
    if (ci<0||di>1)
      {
    cout<<"boundary exceeds"<<endl;
    return 0;
      }
    x=ci;
    x1=x*x;
    y1=abs(ai-x1);
    y2=y1*y1-bi*bi;
    y3=sqrt(y2);
    y=1/y3;//initial vaule
    j=2;
    while(i<numberr-1)
    {
     j=6-j;
     x=ci+i*hi;
     x1=x*x;
```

```
      y1=abs(ai-x1);
      y2=y1*y1-bi*bi;
      y3=sqrt(y2);
      y=y+j/y3;
      i++;
     }
     x=ci+numberr*hi;
     x1=x*x;
     y1=abs(ai-x1);
     y2=y1*y1-bi*bi;
     y3=sqrt(y2);
     y=y+1/y3;
     y=y*hi/3;//end of 1st part
     //second part
     x=0.5*di+0.5*ci+0.5*numberr*hi;
     x1=x*x;
     y1=abs(ai-x1);
     y2=y1*y1-bi*bi;
     y3=sqrt(y2);
     y2=1/y3;
     y=y+y2*(di-ci-numberr*hi);
     //finish the integration
     return y;
   }
}
```

```
#include <iostream>
#include <cmath>
#include <cstdlib>
using namespace std;


double integralsqrt(double a_sqrt,double b_sqrt,
                    double c_sqrt,double d_sqrt)
{
 double stepp=0.00002;
 double hi=stepp/2;
 double rangee=d_sqrt-c_sqrt;
 double numberr=2*floor(rangee/stepp);
 // the range is then divvided into 2 parts.
 //the 1st part will be integrated by simpson's rule,
 // the second part will
 // using mid-point rule
 int i,j;
 i=1;
 double x,x1,y,y1,y2,y3;
 if(rangee<=stepp) return y=0;
 else{//else
   if (c_sqrt <0||d_sqrt >1)
     {//if
       cout<<"boundary exceeds"<<endl;
       return 1;
     }//if
   x=c_sqrt;
   x1=x*x;
   y1=abs(a_sqrt-x1);
   y2=y1*y1-b_sqrt*b_sqrt;
   y3=sqrt(y2);
   y=y3;//initial vaule
   j=2;
   while(i<numberr-1)
     {
   j=6-j;// give the 4,2,4,2.........
   x=c_sqrt+i*hi;
   x1=x*x;
   y1=abs(a_sqrt-x1);
   y2=y1*y1-b_sqrt*b_sqrt;
   y3=sqrt(y2);
   y=y+j*y3;
```

```
      i++;
        }
      x=c_sqrt+numberr*hi;
      x1=x*x;
      y1=abs(a_sqrt-x1);
      y2=y1*y1-b_sqrt*b_sqrt;
      y3=sqrt(y2);
      y=y+y3;
      y=y*hi/3;//end of 1st part
      //second part
      x=0.5*d_sqrt+0.5*c_sqrt+0.5*numberr*hi;
      x1=x*x;
      y1=abs(a_sqrt-x1);
      y2=y1*y1-b_sqrt*b_sqrt;
      y3=sqrt(y2);
      y2=y3;
      y=y+y2*(d_sqrt-c_sqrt-numberr*hi);
      return y;
   }// else
}
```

```
#include <iostream>
#include <cmath>
#include <cstdlib>
#include <fstream>
using namespace std;

double integrallog(double al, double bl,
                   double cl, double dl);
double integralinverse(double ai, double bi,
                       double ci, double di);
double integralinverse2(double a2, double b2,
                        double c2, doubled2);
double integralsqrt(double a_sqrt, double b_sqrt,
                    double c_sqrt, double d_sqrt);
double integralsqrt2(double a_sqrt2, double b_sqrt2,
                     double c_sqrt2, double d_sqrt2);
double intsqrtup(double a_up, double b_up,
                 doublec_up, double d_up);
double intsqrtdown(double a_down, doubleb_down,
                   double c_down, double d_down);
double intsqrtdown2(double a_down2, double b_down2,
                    double c_down2,double d_down2);
double inttripleinverse(double at,double bt,
                        double ct,double dt);
double inttripleinverse2(double a3,double b3,
                         double c3,double d3);

int main(){
  //begin to define variables for physical quantities
  int n=61;
  int m=61;//n*m is the number of lattice points
  int k=1;
  int l=1;//iterative number
  int loop;//loop
  int hnumber;
  double dos=0.2;//N(0)*coupling cosntant
  double homega=0.5*exp(1/dos);
  //Debye freguency in terms of BCS Gap
  double h;//magnetic field
  double sh;//used to find minimum
  double fh;
  double th;
```

```
double qh;
//used to find minimum external magnetic field
double ii=(dos+1)*(dos+1)/(2*dos);//some constant
double deltamu;//the energy difference
double deltamu2;
double deltamu3;
double deltamu4;
double z;// the deform energy
double z2;
double z3;
double z4;
double value_logy1;
double value_logy2;
double value_inversex1;
double value_inversex2;
double value_inversez1;
double value_inversez2;// value used in derived gap
double value_sqrtx1;
double value_sqrtx2;
double value_sqrtz1;
double value_sqrtz2;
double value_up1;
double value_up2;
double value_downx1;
double value_downx2;
double value_downz1;
double value_downz2;
double valuetr1;
double valuetr2;
double value1;
double value2;
double value3;
double value4;
double value5;
double value6;
double value7;
double value8;
double delta2;
double delta1;          // initial value
double left_side;
double right_side;
double derivative2;
double log_of_delta;
double power_of_e;
double energy1;
```

```
double energy2=10;
double energy3=10;
double energy4=10;
//used to find min energy, useless initial value
double delta;
double fdelta;//used to find min energy
double sdelta;
double tdelta;//used to find min energy
double qdelta;
double pass=0;// to find multi solution
double energy;
double boundary_c;
double boundary_d;
double a;// cosntant
double b;//constants and range used in function gap,a,b
double a1;//range
double b1;//range
double stop_condition;
double gapstep;
double compare1;
double compare2;
double compare;
double condition;//useless
double overmu;
double overmu1;
double overmu2;
double overmu3;
double overmu4;
double overz;
double overz1;
double overz2;
double overz3;
double overz4;
double realsolution;
double test,test1,test2,test3,test4;
cout<<"define over"<<endl;

ofstream deformfermi("newer.txt");
ofstream energymin1("mini1.txt");
ofstream energymin2("mini2.txt");
ofstream energymin3("mini3.txt");
ofstream energymin4("mini4.txt");
ofstream sarma("sarma.txt");
if(deformfermi.fail()||energymin1.fail()
```

```
  ||energymin2.fail()||energymin3.fail()
  ||energymin4.fail())
  {
    cerr<<"unable to open the file for writing"<<endl;
    return 1;
  }
else   cout<<"open successful"<<endl;

//start to calculate the delta and energy at
for(hnumber=1;hnumber<=40;hnumber++)
  {
    h=0.0001+0.03*(hnumber-1);
    z2=1000;
    overmu2=1000;
    overz2=1000;
    energy2=1000;
    sh=1000;
    //initial value, 1000 meaningless, can't be reached.
    sdelta=1000;
    deltamu2=1000;
    cout<<hnumber<<endl;
    for(k=1;k<=n;k++)
     {// first for bra
      deltamu=0.0001+0.02*(k-1);//deltamu's value
      for(l=1;l<=m;l++)
       {//second for bra
        // cout<<"l="<<l<<endl;
        z=0.0001+0.05*(l-1);
        //z's value,lattice(deltamu, z);
        pass=0;
        loop=1;
        stop_condition=10;// initial value, useless;
        gapstep=0.1;
        while(loop<=2||stop_condition>=0.0001)
        {//bra for while
         if (loop==1)
           {//ket for while's if

             if(pass==0)
           {
             delta1=0.001;
           }
            delta2=delta1;//back up
```

```
log_of_delta=log(delta1);
boundary_c=(deltamu-delta1)/z;
boundary_d=(deltamu+delta1)/z;
a=deltamu/z;
//constants and range used in function gap
b=delta1/z;
//constants and range used in function gap

if (boundary_c<0) {boundary_c=0;}
if (boundary_c>1){boundary_c=1;}
if (boundary_d>1){boundary_d=1;}
if (boundary_d<0){boundary_d=0;}

if(boundary_c>=0&&boundary_d<=1)
{//1st

 a1=0.999*sqrt(boundary_c);
 b1=1.001*sqrt(boundary_d);
 // those function have integrable
 // singularities on these two boundaries.
 //the 0.999 and 1.001 are
 //used to avoid the singularities.
 value_logy1=integrallog(a,b,0.,a1);
 value_logy2=integrallog(a,b,b1,1.);

 value_inversex1=integralinverse(a,b,0.,a1);
 value_inversex2=integralinverse(a,b,b1,1.);

 value_inversez1=integralinverse2(a,b,0.,a1);
 value_inversez2=integralinverse2(a,b,b1,1.);

 value1=value_logy1+value_logy2;
 value2=(value_inversex1-value_inversex2)/z;
 value3=value_inversez1+value_inversez2;
 valuetr1=inttripleinverse(a,b,0.,a1)
           -inttripleinverse(a,b,b1,1.);
 valuetr2=inttripleinverse2(a,b,0.,a1)
           +inttripleinverse2(a,b,b1,1.);

 left_side=2*h*value2
    +2*(1-dos-2*dos*log_of_delta)*value1
    -2*dos*value1*value1-2*(1
```

```
          -dos*log_of_delta
        - dos*value1)*value3+ii -2;
   right_side=0.5*(1-2*dos*log_of_delta-dos)
              *(1-2*dos*log_of_delta-dos)/dos;
   derivative2=2*(1-2*dos-dos*log_of_delta)
      -dos*2*value1+2*dos*value3
      +2*dos*h*value2+2*dos*h*b*b*valuetr1
      -2*(1-dos*log_of_delta-dos-dos*value1
             +value3)
      *(1-dos*log_of_delta-dos-dos*value1
           +dos*value3)
      -2*dos*(-1+valuetr2)
      *(1-dos*log_of_delta-dos*value1);

  }//1st
  else{//1sttt

         cerr<<"boundary error"<<endl;
          return 1;

        }//1sttt
  compare1=left_side-right_side;
  compare2=compare1;
  delta1=delta1+gapstep;

 }//ket for while's if, for the first loop

else//the following is for the second loop and
 {//for else in while

  log_of_delta=log(delta1);
  boundary_c=(deltamu-delta1)/z;
  boundary_d=(deltamu+delta1)/z;
  a=deltamu/z;
  b=delta1/z;


  if (boundary_c<0) {boundary_c=0;}
  if (boundary_c>1) {boundary_c=1;}
  if (boundary_d>1) {boundary_d=1;}
  if (boundary_d<0) {boundary_d=0;}
  if(boundary_c>=0&&boundary_d<=1)
```

```
{//1st

 a1=0.999*sqrt(boundary_c);
  b1=1.001*sqrt(boundary_d);
  //the 0.999 and 1.001 are used to avoid
  // the singularities.
  value_logy1=integrallog(a,b,0.,a1);
  value_logy2=integrallog(a,b,b1,1.);

  value_inversex1=integralinverse(a,b,0.,a1);
  value_inversex2=integralinverse(a,b,b1,1.);

  value_inversez1=integralinverse2(a,b,0.,a1);
  value_inversez2=integralinverse2(a,b,b1,1.);

  value1=value_logy1+value_logy2;
  value2=(value_inversex1-value_inversex2)/z;
  value3=value_inversez1+value_inversez2;
  valuetr1=inttripleinverse(a,b,0.,a1)
          -inttripleinverse(a,b,b1,1.);
  valuetr2=inttripleinverse2(a,b,0.,a1)
          +inttripleinverse2(a,b,b1,1.);

  left_side=2*h*value2
    +2*(1-dos-2*dos*log_of_delta)*value1
    -2*dos*value1*value1-2*(1-dos*log_of_delta
    - dos*value1)*value3+ii-2;
  right_side=0.5*(1-2*dos*log_of_delta-dos)
          *(1-2*dos*log_of_delta-dos)/dos;
  derivative2=2*(1-2*dos-dos*log_of_delta)
    -dos*2*value1+2*dos*value3
    +2*dos*h*value2+2*dos*h*b*b*valuetr1
    -2*(1-dos*log_of_delta-dos-dos*value1
          +value3)
    *(1-dos*log_of_delta-dos-dos*value1
        +dos*value3)
    -2*dos*(-1+valuetr2)
    *(1-dos*log_of_delta-dos*value1);
}//1st

else{//1sttt
```

```
        cerr<<"boundary error"<<endl;
        return 1;

        }//1sttt
  compare1=left_side-right_side;
  if(compare1*compare2>0)
  {//00if
    delta2=delta1;
    delta1=delta1+gapstep;
    compare2=compare1;
    //discard the original compare2,
    //if the sign are the same
   }//00if

   else if(compare1*compare2<0)
   {// 02 else

    //cout<<"cat3"<<"\t"<<endl;
    delta1=delta2;
     //sign is changed, so
     // keep the original h
    gapstep=(gapstep/10.0);
    //step is changed
    delta1=delta1+gapstep;

    // sign is changed ,
    //so  don't give compare2 a new value
   }// 02 else
   else if(compare1==0)
   {// 04 else
    delta2=delta1;
    //this is  not going to happen
   }// 04 else
   else if(compare2==0)
   {// 05 else
    delta1=delta2;
    //this is not going to happen
    }// 05 else
}//for_else in_while

// this part is used limit the max value
if (delta1>3)
```

```
 {
  delta1=delta2=0;
  deformfermi<<deltamu<<"\t"<<z<<"\t"
             <<"large_delta"<<"\n";
  pass=4;
 }
 else if(delta1<=0.0001)
 {
  delta1=delta2=0.0001;
}
stop_condition=abs(delta1-delta2);
loop++;
compare=abs(compare1);
if(stop_condition<=0.0001&&compare>1&&pass<=2)
{

 delta1=delta1+4*gapstep;//the 4 is used to
                         //avoid the single point
 gapstep=0.1;
 pass++;
 loop=1;
}
else if(stop_condition<=0.0001&&compare<=0.1
          &&delta1<=0.999999&&loop>1&&pass<=2)
{
 condition=1-dos*log_of_delta-dos*value1;
 value_sqrtx1=integralsqrt(a,b,0,a1);
 value_sqrtx2=integralsqrt(a,b,b1,1.0);

 value_sqrtz1=integralsqrt2(a,b,0,a1);
 value_sqrtz2=integralsqrt2(a,b,b1,1.0);

 value4=value_sqrtx1-value_sqrtx2;
 value5=value_sqrtz1+value_sqrtz2;

 value_up1=intsqrtup(a,b,0,a1);
 value_up2=intsqrtup(a,b,b1,1);
 value6=value_up1-value_up2;

 value_downx1=intsqrtdown(a,b,0,a1);
 value_downx2=intsqrtdown(a,b,b1,1);
 value7=(value_downx1-value_downx2)/z;
```

```
value_downz1=intsqrtdown2(a,b,0,a1);
value_downz2=intsqrtdown2(a,b,b1,1);
value8=value_downz1+value_downz2;

overz=-z*value6+h*value8
      -delta1*delta1*(1-
        n*log_of_delta-n*value1)*value7;
overmu=2*value4-2*h*value3
          -2*delta1*delta1*(1-
           dos*log_of_delta-dos*value1)*(-value2);
energy=delta1*delta1*(1-dos*log_of_delta)
       -0.5*dos*delta1*delta1
      -dos*delta1*delta1*value1+dos*z*z*value5
      -2*dos*h*z*value4
      -(delta1-dos*delta1*log_of_delta
           -dos*delta1*value1)
      *(delta1-dos*delta1*log_of_delta
           -dos*delta1*value1);
deformfermi<<deltamu<<"\t"<<z<<"\t"
           <<delta1<<"\t"<<"\t"
           <<energy<<"\t"<<"\t"
           <<value1<<"\t"<<"\t"<<value2
           <<"\t"<<"\t"
           <<value3<<"\t"<<overmu<<"\t"
           <<derivative2<<"\n";
if(h==deltamu&&deltamu>delta1&&z<0.05)
{
 sarma<<h<<"\t"<<delta1<<"\t"<<0-energy
      <<"\t"<<overz<<"\t"<<overmu<<"\t"
      <<derivative2<<"\n";
}
if(boundary_c==0&&boundary_d==1)//  four cases
{
 energy1=energy;
 fdelta=delta1;
 fh=h;
 energymin1<<fh<<"\t"<<dos*fh*fh<<"\t"<<deltamu
           <<"\t"<<z<<"\t"<<delta1<<"\t"
           <<0-energy1<<"\t"<<overz<"\t"
           <<overmu<<"\n";
}
else if(energy<=energy2&&deltamu<delta
```

```
          &&boundary_d<1)
{
 energy2=energy;
 sdelta=delta1;
 sh=h;
 overmu2=overmu;
 overz2=overz;
 deltamu2=deltamu;
 z2=z;
}
else if(energy<=energy3&&deltamu>=delta
          &&boundary_d==1)
{
  energy3=energy;
  tdelta=delta1;
  th=h;
  overmu3=overmu;
  overz3=overz;
  deltamu3=deltamu;
  z3=z;
 }
else if(energy<=energy4&&deltamu>delta
          &&boundary_d<1)
 {
  energy4=energy;
  qdelta=delta1;
  qh=h;
  overmu4=overmu;
  overz4=overz;
  deltamu4=deltamu;
  z4=z;
 }
 deformfermi<<deltamu<<"\t"<<z<<"\t"
               <<delta1<<"\t"
               <<"\t"<<0-energy
               <<"\t"<<"\t"<<overz<<"\t"
               <<overmu<<"\t"
               <<derivative2<<"\n";
  gapstep=0.1;
 pass++;
  loop=1;
  delta1=delta1+2*gapstep;
 }
```

```
  else if (pass>=3&&compare>1)
  {
   deformfermi<<deltamu<<"\t"<<z<<"\t"
              <<"no_solution"
              <<"compare"<<left_side-right_side<<"\n";
  }
}//ket for while

deformfermi<<deltamu<<"\t"<<z<<"\t"<<delta1
          <<"\t"<<"\n";
if(compare1<=0.1&&compare1>=-0.1&&pass<=2)
{
 condition=1-dos*log_of_delta-dos*value1;
 value_sqrtx1=integralsqrt(a,b,0,a1);
 value_sqrtx2=integralsqrt(a,b,b1,1.0);

 value_sqrtz1=integralsqrt2(a,b,0,a1);
 value_sqrtz2=integralsqrt2(a,b,b1,1.0);

 value4=value_sqrtx1-value_sqrtx2;
 value5=value_sqrtz1+value_sqrtz2;

         value_up1=intsqrtup(a,b,0,a1);
 value_up2=intsqrtup(a,b,b1,1);
 value6=value_up1-value_up2;

 value_downx1=intsqrtdown(a,b,0,a1);
 value_downx2=intsqrtdown(a,b,b1,1);
 value7=(value_downx1-value_downx2)/z;

 value_downz1=intsqrtdown2(a,b,0,a1);
 value_downz2=intsqrtdown2(a,b,b1,1);
 value8=value_downz1+value_downz2;

 overz=-z*value6+h*value8
       -delta1*delta1*(1-
            dos*log_of_delta-n*value1)*value7;
 overmu=2*value4-2*h*value3
        +2*delta1*delta1*(1-
            dos*log_of_delta-dos*value1)*(-value2);
 energy=delta1*delta1*(1-dos*log_of_delta)
        -0.5*dos*delta1*delta1
        -dos*delta1*delta1*value1+dos*z*z*value5
```

```
        -2*dos*h*z*value4
        -(delta1-dos*delta1*log_of_delta
            -dos*delta1*value1)
        *(delta1-dos*delta1*log_of_delta
            -dos*delta1*value1);

if(h==deltamu&&h>deltamu&&z<0.05)
  {
      sarma<<h<<"\t"<<delta1<<"\t"
          <<0-energy<<"\t"<<overz
          <<"\t"<<overmu<<"\t"<<derivative2<<"\n";
  }
if(deltamu<delta1&&boundary_d==1)
  {
     energy1=energy;
     fdelta=delta1;
     fh=h;
     energymin1<<fh<<"\t"<<fh*fh*dos<<"\t"<<deltamu
             <<"\t"<<z
             <<"\t"<<delta1<<"\t"<<0-energy
             <<"\t"<<overz
             <<"\t"<<overmu<<"\n";
  }
else if(energy<=energy2&&deltamu<delta
        &&boundary_d<1)
  {
     energy2=energy;
     sdelta=delta1;
     sh=h;
     overmu2=overmu;
     overz2=overz;
     deltamu2=deltamu;
     z2=z;
  }
else if(energy<=energy3&&deltamu>=delta
        &&boundary_d==1)
  {
     energy3=energy;
     tdelta=delta1;
     th=h;
     overmu3=overmu;
     overz3=overz;
     deltamu3=deltamu;
```

```
            z3=z;
          }
         else if(energy<=energy4&&deltamu>delta
                &&boundary_d<1)
          {
            energy4=energy;
            qdelta=delta1;
            qh=h;
            overmu4=overmu;
            overz4=overz;
            deltamu4=deltamu;
            z4=z;
          }

            deformfermi<<deltamu<<"\t"<<z<<"\t"
                       <<delta1<<"\t"
                       <<"\t"<<0-energy<<"\t"<<"\t"
                       <<overz<<"\t"
                       <<overmu<<"\t"<<derivative2<<"\n";
        }
      }//second for bra

   } //first for bra

   energymin2<<sh<<"\t"<<deltamu2<<"\t"<<z2
            <<"\t"<<sdelta<<"\t"
            <<0-energy2<<"\t"<<overz2<<"\t"
            <<overmu2<<"\t"
            <<derivative2<<"\n";
   energymin3<<th<<"\t"<<deltamu3<<"\t"<<z3<<"\t"
            <<tdelta<<"\t"
            <<0-energy3<<"\t"<<overz3<<"\t"
            <<overmu3<<"\t"
            <<derivative2<<"\n";
   energymin4<<qh<<"\t"<<deltamu4<<"\t"<<z4<<"\t"
            <<qdelta<<"\t"
            <<0-energy4<<"\t"<<overz4<<"\t"
            <<overmu4<<"\t"
            <<derivative2<<"\n";
}//for h
deformfermi<<endl;
energymin1<<endl;
```

```
  energymin2<<endl;
  energymin3<<endl;
  energymin4<<endl;
  return 0;
}// for main
```

C. Details in Chapters II and III

$$\begin{aligned}\mathcal{H} &= \sum_{\vec{k}}[\frac{\hbar^2 k^2}{2m_1} - \mu_1(1-\epsilon_1\cos^2\theta)]c^\dagger_{k\uparrow}c_{k\uparrow} \\ &+ \sum_{-\vec{k}}[\frac{\hbar^2 k^2}{2m_2} - \mu_2(1-\epsilon_2\cos^2\theta)]c^\dagger_{-k\downarrow}c_{-k\downarrow} - \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}c_{-k\downarrow}c_{k\uparrow} \qquad \text{(A.3)}\end{aligned}$$

by using self-consistent mean field theory, the above Hamiltonain can be written as:

$$\begin{aligned}\mathcal{H}\prime &\simeq \sum_{\vec{k}}[\frac{\hbar^2 k^2}{2m_1} - \mu_1(1-\epsilon_1\cos^2\theta)]c^\dagger_{k\uparrow}c_{k\uparrow} \\ &+ \sum_{-\vec{k}}[\frac{\hbar^2 k^2}{2m_2} - \mu_2(1-\epsilon_2\cos^2\theta)]c^\dagger_{-k\downarrow}c_{-k\downarrow} - \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}\langle c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}\rangle c_{-k\downarrow}c_{k\uparrow} \\ &- \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}\langle c_{-k\downarrow}c_{k\uparrow}\rangle \\ &+ \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}\langle c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}\rangle\langle c_{-k\downarrow}c_{k\uparrow}\rangle \qquad \text{(A.4)}\end{aligned}$$

In order to simplify calculation, we let

$$a_k = \frac{\hbar^2 k^2}{2m_1} - \mu_1(1-\epsilon_1\cos^2\theta); \qquad \text{(A.5)}$$

$$b_k = \frac{\hbar^2 k^2}{2m_2} - \mu_2(1-\epsilon_2\cos^2\theta); \qquad \text{(A.6)}$$

$$\Delta_{k\prime} = \sum_{\vec{k}} V_{k,k\prime}\langle c_{-k\downarrow}c_{k\uparrow}\rangle \qquad \text{(A.7)}$$

Then we obtain:

$$\begin{aligned}\mathcal{H}\prime \;\simeq\;& \sum_{\vec{k}} a_k c^{\dagger}_{k\uparrow} c_{k\uparrow} + \sum_{-k} b_k c^{\dagger}_{-k\downarrow} c_{-k\downarrow} \\ & - \sum_{\vec{k}} \Delta c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} - \sum_{\vec{k}} \Delta c_{-k\downarrow} c_{k\uparrow} - \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime} \langle c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} \rangle \langle c_{-k\downarrow} c_{k\uparrow} \rangle \end{aligned} \tag{A.8}$$

For S-wave superconductor, we assume $V_{k,k\prime} = V = const$,then

$$\begin{aligned}\mathcal{H}\prime \;\simeq\;& \sum_{\vec{k}} a_k c^{\dagger}_{k\uparrow} c_{k\uparrow} + \sum_{-\vec{k}} b_k c^{\dagger}_{-k\downarrow} c_{-k\downarrow} - \sum_{\vec{k}} \Delta c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} - \sum_{\vec{k}} \Delta c_{-k\downarrow} c_{k\uparrow} + \frac{\Delta^2}{V} \\ =& \sum_{\vec{k}} \begin{pmatrix} c^{\dagger}_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \begin{pmatrix} a_k & -\Delta \\ -\Delta & -b_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{\dagger}_{-k\downarrow} \end{pmatrix} + \sum_{\vec{k}} b_k + \frac{\Delta^2}{V} \end{aligned} \tag{A.9}$$

The eigenvalues of above matrix are

$$\lambda_{k1,k2} = \frac{a_k - b_k}{2} \pm \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2} \tag{A.10}$$

$$u_k^2 = \frac{1}{2}\left(1 + \frac{\frac{a_k+b_k}{2}}{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2}}\right) \tag{A.11}$$

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\frac{a_k+b_k}{2}}{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2}}\right) \tag{A.12}$$

$$\frac{u_k}{v_k} = \frac{\Delta}{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2} - \frac{a_k-b_k}{2}} = \frac{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2} + \frac{a_k-b_k}{2}}{\Delta} \tag{A.13}$$

Assume $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\begin{pmatrix} a_3 \\ a_4 \end{pmatrix}$ are the eigenvalues, then we obtain:

$$\begin{pmatrix} a_k - \lambda_{k1} & -\Delta \\ -\Delta & -b_k - \lambda_{k1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \tag{A.14}$$

or

$$\begin{pmatrix} a_k & -\Delta \\ -\Delta & -b_k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \lambda_{k1} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

and

$$\begin{pmatrix} a_k - \lambda_{k2} & -\Delta \\ -\Delta & -b_k - \lambda_{k2} \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = 0$$

or

$$\begin{pmatrix} a_k & -\Delta \\ -\Delta & -b_k \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \lambda_{k2} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} \tag{A.15}$$

Then we obtain:

$$\frac{a_1}{a_2} = \frac{\Delta}{-\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2} + \frac{a_k-b_k}{2}} = \frac{u_k}{-v_k} \tag{A.16}$$

$$\frac{a_3}{a_4} = \frac{\Delta}{\sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2} + \frac{a_k-b_k}{2}} = \frac{v_k}{u_k} \tag{A.17}$$

and the transformation matrix:

$$\mathbf{T} = \begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix} \quad and \quad \mathbf{T}^\dagger\mathbf{T} = I \tag{A.18}$$

$$\mathbf{T}^\dagger \begin{pmatrix} a_k & -\Delta \\ -\Delta & -b_k \end{pmatrix} \mathbf{T} = \mathbf{T}^\dagger \begin{pmatrix} \lambda_{k1} u_k & \lambda_{k2} v_k \\ -\lambda_{k1} v_k & \lambda_{k2} u_k \end{pmatrix} = \begin{pmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{pmatrix} \tag{A.19}$$

By using the matrix, the Hamiltonian can be written as:

$$\begin{aligned} \mathcal{H}' &= \sum_{\vec{k}} \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \mathbf{T}\mathbf{T}^\dagger \begin{pmatrix} a_k & -\Delta \\ -\Delta & -b_k \end{pmatrix} \mathbf{T}\mathbf{T}^\dagger \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + \sum_{\vec{k}} b_k + \frac{\Delta^2}{V} \\ &= \sum_{\vec{k}} \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \mathbf{T} \begin{pmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{pmatrix} \mathbf{T}^\dagger \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + \sum_{\vec{k}} b_k + \frac{\Delta^2}{V} \\ &= \sum_{\vec{k}} \begin{pmatrix} \alpha^\dagger_k & \beta_k \end{pmatrix} \begin{pmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta^\dagger_k \end{pmatrix} + \sum_{\vec{k}} b_k + \frac{\Delta^2}{V} \\ &= \sum_{\vec{k}} \left(\lambda_{k1} \alpha^\dagger_k \alpha_k + \lambda_{k2} \beta_k \beta^\dagger_k \right) + \sum_{\vec{k}} b_k + \frac{\Delta^2}{V} \end{aligned} \tag{A.20}$$

where

$$\begin{pmatrix} \alpha_k \\ \beta^\dagger_k \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \tag{A.21}$$

Now, let's consider $\lambda_{k1}\alpha^\dagger_k\alpha_k$. When $\lambda_{k1} < 0$, the number operator $\alpha^\dagger_k\alpha_k$ will favor 1 rather than 0, which makes the Hamiltonian looks like unreasonable because no quasiparticle should be excited at ground state. Then we should use $(1 - \alpha_k\alpha^\dagger_k)$

instead of $\alpha_k^\dagger \alpha_k$ and redefine the annihilation and creation operators. So we have three cases here:

Case 1:$\lambda_{k1,k2} > 0$

$$\mathcal{H}' = \sum_{\vec{k}} \left(\lambda_{k1} \alpha_k^\dagger \alpha_k + \lambda_{k2} \beta_k \beta_k^\dagger \right) + \sum_{\vec{k}} b_k + \frac{\Delta^2}{V} \tag{A.22}$$

Case 2:$\lambda_{k1} > 0$ while $\lambda_{k2} < 0$

$$\mathcal{H}' = \sum_{\vec{k}} \left(\lambda_{k1} \alpha_k^\dagger \alpha_k - \lambda_{k2} \beta_k^\dagger \beta_k \right) + \sum_{\vec{k}} (b_k + \lambda_{k2}) + \frac{\Delta^2}{V} \tag{A.23}$$

Case 3:$\lambda_{k1,k2} < 0$

$$\begin{aligned} \mathcal{H}' &= \sum_{\vec{k}} \left(-\lambda_{k1} \alpha_k \alpha_k^\dagger - \lambda_{k2} \beta_k^\dagger \beta_k \right) + \sum_{\vec{k}} (b_k + \lambda_{k1} + \lambda_{k2}) + \frac{\Delta^2}{V} \\ &= \sum_{\vec{k}} \left(-\lambda_{k1} \alpha_k \alpha_k^\dagger - \lambda_{k2} \beta_k^\dagger \beta_k \right) + \sum_{\vec{k}} a_k + \frac{\Delta^2}{V} \end{aligned} \tag{A.24}$$

Then let's consider case 1: $\lambda_{k1,k2} > 0$:

$$\frac{a_k - b_k}{2} > 0 \quad and \quad \frac{a_k - b_k}{2} \geq \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2} \tag{A.25}$$

$\frac{a_k - b_k}{2} > 0$:

$$a_k - b_k > 0$$

$$\frac{\hbar^2 k^2}{2m} - \mu_1(1 - \epsilon_1 \cos^2\theta) - [\frac{\hbar^2 k^2}{2m} - \mu_2(1 - \epsilon_2 \cos^2\theta)] > 0$$

$$(\mu_1 \epsilon_1 - \mu_2 \epsilon_2) \cos^2\theta > \mu_1 - \mu_2$$

$$z\cos^2\theta > \delta\mu$$
$$\cos^2\theta > \frac{\delta\mu}{z}$$

where

$$z = \frac{\mu_1\epsilon_1 - \mu_2\epsilon_2}{2} \tag{A.26}$$
$$\delta\mu = \frac{\mu_1 - \mu_2}{2} \tag{A.27}$$

$\frac{a_k - b_k}{2} \geq \sqrt{(\frac{a_k+b_k}{2})^2 + \Delta^2}$:

$$a_k b_k + \Delta^2 \leq 0$$
$$(\frac{\hbar^2 k^2}{2m} - c)(\frac{\hbar^2 k^2}{2m} - d) + \Delta^2 \leq 0$$
$$\frac{c+d}{2} - \sqrt{(\frac{c-d}{2})^2 - \Delta^2} \leq \frac{\hbar^2 k^2}{2m} \leq \frac{c+d}{2} + \sqrt{(\frac{c-d}{2})^2 - \Delta^2}$$
$$\mu - \alpha(\theta) - \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2} \leq \frac{\hbar^2 k^2}{2m} \leq \mu - \alpha(\theta) + \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2}$$
$$-\alpha(\theta) - \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2} \leq \frac{\hbar^2 k^2}{2m} - \mu \leq -\alpha(\theta) + \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2}$$
$$\epsilon_- \quad \leq \quad \epsilon_k \quad \leq \quad \epsilon_+$$

where

$$\begin{aligned}
c &= \mu_1(1-\epsilon_1\cos^2\theta) \\
d &= \mu_2(1-\epsilon_2\cos^2\theta) \\
\alpha(\theta) &= \frac{\mu_1\epsilon_1+\mu_2\epsilon_2}{2}\cos^2\theta \\
\mu &= \frac{\mu_1+\mu_2}{2} \\
\epsilon_k &= \frac{\hbar^2k^2}{2m}-\mu \\
\epsilon_\pm &= -\alpha \pm \sqrt{(\delta\mu - z\cos^2\theta)^2-\Delta^2}
\end{aligned}$$

However, $\epsilon_\pm$ exists only when

$$(\delta\mu - z\cos^2\theta)^2-\Delta^2 \geq 0$$

$$0 \leq \cos^2\theta \leq \frac{\delta\mu-\Delta}{z} \quad or \quad \frac{\delta\mu+\Delta}{z} \leq \cos^2\theta \leq 1$$

So case 1 ($\lambda_{k1,k2} > 0$) exists in the following region:

$$\epsilon_- \leq \epsilon_k \leq \epsilon_+$$

$$\frac{\delta\mu+\Delta}{z} \leq \cos^2\theta \leq 1 \tag{A.28}$$

Let's call this region D_2. While $\frac{\delta\mu}{z} \leq \cos^2\theta \leq \frac{\delta\mu+\Delta}{z}$, $\frac{a_k-b_k}{2} \geq \sqrt{(\frac{a_k+b_k}{2})^2+\Delta^2}$ does not have any solution, so within this region $\lambda_{k1} \geq 0, \lambda_{k2} \leq 0$.

Then lets consider case 3 before case 2:

$$\lambda_{k1,k2} < 0$$

or in other words:

$$\frac{a_k - b_k}{2} < 0 \quad and \quad -\frac{(a_k - b_k)}{2} \geq \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2}$$

$\frac{a_k - b_k}{2} < 0$:

$$a_k - b_k < 0$$

$$\frac{\hbar^2 k^2}{2m} - \mu_1(1 - \epsilon_1 \cos^2\theta) - [\frac{\hbar^2 k^2}{2m} - \mu_2(1 - \epsilon_2 \cos^2\theta)] < 0$$

$$(\mu_1\epsilon_1 - \mu_2\epsilon_2)\cos^2\theta < \mu_1 - \mu_2$$

$$z\cos^2\theta < \delta\mu$$

$$\cos^2\theta < \frac{\delta\mu}{z}$$

and $-\frac{a_k - b_k}{2} \geq \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2}$:

$$a_k b_k + \Delta^2 \leq 0$$

$$(\frac{\hbar^2 k^2}{2m} - c)(\frac{\hbar^2 k^2}{2m} - d) + \Delta^2 \leq 0$$

$$\frac{c+d}{2} - \sqrt{(\frac{c-d}{2})^2 - \Delta^2} \leq \frac{\hbar^2 k^2}{2m} \leq \frac{c+d}{2} + \sqrt{(\frac{c-d}{2})^2 - \Delta^2}$$

$$\mu - \alpha(\theta) - \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2} \leq \frac{\hbar^2 k^2}{2m} \leq \mu - \alpha(\theta) + \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2}$$

$$-\alpha(\theta) - \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2} \leq \frac{\hbar^2 k^2}{2m} - \mu \leq -\alpha(\theta) + \sqrt{(\delta\mu - z\cos^2\theta)^2 - \Delta^2}$$

$$\epsilon_- \leq \epsilon_k \quad \leq \quad \epsilon_+$$

And $\epsilon_\pm$ exists only when

$$0 \leq \cos^2\theta \leq \frac{\delta\mu - \Delta}{z} \quad or \quad \frac{\delta\mu + \Delta}{z} \leq \cos^2\theta \leq 1$$

So case $3(\lambda_{k1,k2} < 0)$ exists in the following region:

$$\epsilon_- \leq \epsilon_k \leq \epsilon_+$$

$$0 \leq \cos^2\theta \leq \frac{\delta\mu - \Delta}{z} \tag{A.29}$$

Let's call this region D_1. Actually, there are two regions: one for spin up, the other is for spin down, but only one of them is occupied by electrons. When $\frac{\delta\mu-\Delta}{z} \leq \cos^2\theta \leq \frac{\delta\mu}{z}$, $\epsilon_\pm$ does not have any solution, so within this region $\lambda_{k1} \geq 0, \lambda_{k2} \leq 0$.

Case 2:

$$\begin{aligned}
&\lambda_{1k} \geq 0 \quad and \quad \lambda_{2k} \leq 0;\\
&in \quad other \quad word:\\
&when \quad \frac{a_k - b_k}{2} < 0; \quad -\frac{a_k - b_k}{2} \leq \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2}\\
&when \quad \frac{a_k - b_k}{2} > 0; \quad \frac{a_k - b_k}{2} \leq \sqrt{(\frac{a_k + b_k}{2})^2 + \Delta^2}
\end{aligned} \tag{A.30}$$

So when $\epsilon_k \leq \epsilon_-$ or $\epsilon_k \geq \epsilon_+$ and the angle satisfies: $0 \leq \cos^2\theta \leq \frac{\delta\mu-\Delta}{z}$ or $\frac{\delta\mu+\Delta}{z} \leq \cos^2\theta \leq 1$, we have $\lambda_{1k} \geq 0$ and $\lambda_{2k} \leq 0$; when $\frac{\delta\mu-\Delta}{z} \leq \cos^2\theta \leq \frac{\delta\mu+\Delta}{z}$, we always have $\lambda_{1k} \geq 0$ and $\lambda_{2k} \leq 0$. That's the region other than D_1 and D_2, let's call this region D.

In conclusion, in region D, we have pairings and the Hamiltonian is:

$$\mathcal{H}' = \sum_{\vec{k}} \left(\lambda_{k1} \alpha_k^\dagger \alpha_k - \lambda_{k2} \beta_k^\dagger \beta_k \right) + \sum_{\vec{k}} (b_k + \lambda_{k2}) + \frac{\Delta^2}{V} \tag{A.31}$$

in region D_1, we have depairing and the hamiltonian is:

$$\mathcal{H}' = \sum_{\vec{k}} \left(-\lambda_{k1} \alpha_k \alpha_k^\dagger - \lambda_{k2} \beta_k^\dagger \beta_k \right) + \sum_{\vec{k}} a_k + \frac{\Delta^2}{V} \tag{A.32}$$

in region D_2, we have depairing and the hamiltonian is:

$$\mathcal{H}' = \sum_{\vec{k}} \left(\lambda_{k1} \alpha_k^\dagger \alpha_k + \lambda_{k2} \beta_k \beta_k^\dagger \right) + \sum_{\vec{k}} b_k + \frac{\Delta^2}{V} \tag{A.33}$$

The trial ground state wave function of this superconducting state is:

$$|\Psi\rangle = \prod_{D,\vec{k}} (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) \prod_{D_1,\vec{k}} c^\dagger_{k\uparrow} \prod_{D_2,\vec{k}} c^\dagger_{-k\downarrow} |vac\rangle$$

then the particle number for this state is:

$$\begin{aligned}
\langle N\rangle_s &= \langle\Psi| c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{-k\downarrow} c_{-k\downarrow} |\Psi\rangle \\
&= \sum_{D,\vec{k}} \langle vac| (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})^* c^\dagger_{k\uparrow} c_{k\uparrow} ((u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) |vac\rangle \\
&+ \sum_{D,\vec{k}} \langle vac| (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})^* c^\dagger_{-k\downarrow} c_{-k\downarrow} ((u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) |vac\rangle \\
&+ \sum_{D_1,\vec{k}} \langle vac| c_{k\uparrow} c^\dagger_{k\uparrow} c_{k\uparrow} c^\dagger_{k\uparrow} |vac\rangle + \sum_{D_2,\vec{k}} \langle vac| c_{-k\downarrow} c^\dagger_{-k\downarrow} c_{-k\downarrow} c^\dagger_{-k\downarrow} |vac\rangle \\
&= \sum_{D,\vec{k}} 2v_k^2 + \sum_{D_1,\vec{k}} 1 + \sum_{D_2,\vec{k}} 1
\end{aligned}$$

The particle number for the related normal state is :

$$\langle N\rangle_{normal} = \sum_{\vec{k}\uparrow} \theta(k_1 - k) + \sum_{-\vec{k}\downarrow} \theta(k_2 - k)$$

where $\frac{\hbar^2 k_1^2}{2m} = \mu' + h$, $\frac{\hbar^2 k_2^2}{2m} = \mu' - h$, and μ' is the chemical potential of Pauli-Paramagnetism of normal state.

Since the particles far below the fermi surface will not be affected when this metal has a phase transition, we can use the weak coupling approximation when calculating the difference between $\langle N\rangle_s$ and $\langle N\rangle_n$, i.e. the number of particles which locate far below the fermi surface will be canceled. And we also assume the shift of chemical potential should be small when compared with with $\hbar\omega_D$. Then we transform the sum to an integral under the weak coupling approximation as following:

$$\begin{aligned}
\sum_{\vec{k}} &= \frac{V}{(2\pi)^3}\iiint k^2 \sin\theta \, \mathrm{d}k \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \frac{Vk_F}{(2\pi)^2}\int k \, \mathrm{d}k \int \mathrm{d}\cos\theta \\
&= \frac{Vk_F m}{(2\pi\hbar)^2}\int \mathrm{d}\frac{\hbar^2 k^2}{2m}\int \mathrm{d}\cos\theta
\end{aligned}$$

Plug the above equation into $\langle N\rangle_s - \langle N\rangle_n$:

$$\begin{aligned}
\langle N\rangle_s - \langle N\rangle_n &= \frac{Vk_F m}{(2\pi\hbar)^2}\int_D \mathrm{d}\frac{\hbar^2 k^2}{2m}\int_D \mathrm{d}\cos\theta \quad 2v_k^2 \\
&+\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2 k^2}{2m}\int_{D_1} \mathrm{d}\cos\theta + \frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2 k^2}{2m}\int_{D_2} \mathrm{d}\cos\theta \\
&-\frac{Vk_F\prime m}{(2\pi\hbar)^2}\int \mathrm{d}\frac{\hbar^2 k^2}{2m}\int \mathrm{d}\cos\theta \quad \theta(k_1 - k) \\
&-\frac{Vk_F\prime m}{(2\pi\hbar)^2}\int \mathrm{d}\frac{\hbar^2 k^2}{2m}\int \mathrm{d}\cos\theta \quad \theta(k_2 - k)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{Vk_F m}{(2\pi\hbar)^2}\int_D \mathrm{d}\frac{\hbar^2k^2}{2m}\int_D \mathrm{d}\cos\theta \quad 2v_k^2 \\
& +\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1} \mathrm{d}\cos\theta + \frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_2} \mathrm{d}\cos\theta \\
& -\frac{Vk_F\prime m}{(2\pi\hbar)^2}\int_{\mu-\hbar\omega_D}^{\mu+\hbar\omega_D} \mathrm{d}\frac{\hbar^2k^2}{2m}\int \mathrm{d}\cos\theta \quad \theta(k_1-k) \\
& -\frac{Vk_F\prime m}{(2\pi\hbar)^2}\int_{\mu-\hbar\omega_D}^{\mu+\hbar\omega_D} \mathrm{d}\frac{\hbar^2k^2}{2m}\int \mathrm{d}\cos\theta \quad \theta(k_2-k) \\
= \quad & \frac{Vk_F m}{(2\pi\hbar)^2}\int_D \mathrm{d}\frac{\hbar^2k^2}{2m}\int_D \mathrm{d}\cos\theta \quad 2v_k^2 \\
& +\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1} \mathrm{d}\cos\theta + \frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_2} \mathrm{d}\cos\theta \\
& -\frac{2Vk_F\prime m}{(2\pi\hbar)^2}[\frac{\hbar^2k_1^2}{2m}-(\mu-\hbar\omega_D)] - \frac{2Vk_F\prime m}{(2\pi\hbar)^2}[\frac{\hbar^2k_2^2}{2m}-(\mu-\hbar\omega_D)] \\
= \quad & \frac{Vk_F m}{(2\pi\hbar)^2}\int_D \mathrm{d}\frac{\hbar^2k^2}{2m}\int_D \mathrm{d}\cos\theta \quad 2v_k^2 \\
& +\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1} \mathrm{d}\cos\theta + \frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_2} \mathrm{d}\cos\theta \\
& -\frac{Vk_F\prime m}{(2\pi\hbar)^2}[4\mu\prime-4\mu+4\hbar\omega_D]
\end{aligned}
$$

where $\mu\prime = \frac{\hbar^2 k_F\prime^2}{2m}$ is the fermi energy of normal state, and we also used these equations:

$$
\frac{\hbar^2k_1^2}{2m} = \mu\prime + \delta\mu\prime, \quad \frac{\hbar^2k_2^2}{2m} = \mu\prime - \delta\mu\prime
$$

Then let's consider the first term:

$$
\begin{aligned}
1st\,term &= \frac{Vk_F m}{(2\pi\hbar)^2}\int_{-h\omega_D}^{h\omega_D+\Delta} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{-1}^{1}\mathrm{d}\cos\theta \quad (1-\frac{\epsilon_k+\alpha}{\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}}) \\
&\quad -\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1+D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1+D_2}\mathrm{d}\cos\theta \quad (1-\frac{\epsilon_k+\alpha}{\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}}) \\
&= \frac{Vk_F m}{(2\pi\hbar)^2}\int_{-h\omega_D}^{h\omega_D} \mathrm{d}\epsilon_k\int_{-1}^{1}\mathrm{d}\cos\theta \quad (1-\frac{\epsilon_k+\alpha}{\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}}) \\
&\quad +\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1+D_2} \mathrm{d}\epsilon_k\int_{D_1+D_2}\mathrm{d}\cos\theta \quad \frac{\epsilon_k+\alpha}{\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}} \\
&\quad -\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1+D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1+D_2}\mathrm{d}\cos\theta \quad 1 \\
&= \frac{Vk_F m}{(2\pi\hbar)^2}\int_{-1}^{1}\mathrm{d}\cos\theta \quad (\epsilon_k-\sqrt{(\epsilon_k+\alpha)^2+\Delta^2})|_{-\hbar\omega_D}^{\hbar\omega_D} \\
&\quad +\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1+D_2}\mathrm{d}\cos\theta \quad \sqrt{(\epsilon_k+\alpha)^2+\Delta^2}|_{\epsilon_-}^{\epsilon_+} \\
&\quad -\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1+D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1+D_2}\mathrm{d}\cos\theta \quad 1 \\
&= \frac{Vk_F m}{(2\pi\hbar)^2}\int_{-1}^{1}\mathrm{d}\cos\theta \quad (\epsilon_k-\sqrt{(\epsilon_k+\alpha)^2+\Delta^2})|_{-\hbar\omega_D}^{\hbar\omega_D} \\
&\quad -\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1+D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1+D_2}\mathrm{d}\cos\theta \quad 1 \\
&= \frac{Vk_F m}{(2\pi\hbar)^2}\int_{-1}^{1}\mathrm{d}\cos\theta \quad \Big(\hbar\omega_D-\sqrt{(\hbar\omega_D+\alpha)^2+\Delta^2}+\hbar\omega_D \\
&\quad +\sqrt{(-\hbar\omega_D+\alpha)^2+\Delta^2}\Big) \\
&\quad -\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1}\mathrm{d}\cos\theta \quad 1 \\
&\quad -\frac{Vk_F m}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_2}\mathrm{d}\cos\theta \quad 1
\end{aligned}
$$

By omitting the small terms, we have:

$$
\begin{aligned}
= \quad & \frac{Vk_Fm}{(2\pi\hbar)^2}\int_{-1}^{1} \mathrm{d}\cos\theta \quad [2\hbar\omega_D - 2\alpha + \bigcirc(\frac{\Delta^2\alpha}{\hbar^2\omega_D^2})] \\
& -\frac{Vk_Fm}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1} \mathrm{d}\cos\theta \quad 1 \\
& -\frac{Vk_Fm}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_2} \mathrm{d}\cos\theta \quad 1 \\
= \quad & \frac{Vk_Fm}{(2\pi\hbar)^2}\int_{-1}^{1} \mathrm{d}\cos\theta \quad [2\hbar\omega_D - (\mu_1\epsilon_1 + \mu_2\epsilon_2)\cos^2\theta)] \\
& -\frac{Vk_Fm}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1} \mathrm{d}\cos\theta \quad 1 \\
& -\frac{Vk_Fm}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_2} \mathrm{d}\cos\theta \quad 1 \\
= \quad & \frac{Vk_Fm}{(2\pi\hbar)^2}[4\hbar\omega_D - \frac{2}{3}(\mu_1\epsilon_1 + \mu_2\epsilon_2)] \\
& -\frac{Vk_Fm}{(2\pi\hbar)^2}\int_{D_1} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_1} \mathrm{d}\cos\theta - \frac{Vk_Fm}{(2\pi\hbar)^2}\int_{D_2} \mathrm{d}\frac{\hbar^2k^2}{2m}\int_{D_2} \mathrm{d}\cos\theta
\end{aligned}
$$

By putting this 1st term back and define $\Delta k_F = k_F - k_F'$ and $\Delta\mu = \mu - \mu'$, we obtain:

$$
\begin{aligned}
\langle N\rangle_s - \langle N\rangle_n &= \frac{Vk_Fm}{(2\pi\hbar)^2}[4\hbar\omega_D - \frac{2}{3}(\mu_1\epsilon_1 + \mu_2\epsilon_2)] - \frac{Vk_F\prime m}{(2\pi\hbar)^2}[4\mu\prime - 4\mu + 4\hbar\omega_D] \\
&= \frac{4Vm}{(2\pi\hbar)^2}\hbar\omega_D\Delta k_F - \frac{2}{3}\frac{Vk_Fm}{(2\pi\hbar)^2}(\mu_1\epsilon_1 + \mu_2\epsilon_2) + \frac{4Vk_Fm}{(2\pi\hbar)^2}\frac{\hbar^2k_F}{m}\Delta k_F \\
&= \frac{Vm}{(2\pi\hbar)^2}[8\mu\Delta k_F + 4\hbar\omega_D\Delta k_F - \frac{2}{3}(\mu_1\epsilon_1 + \mu_2\epsilon_2)k_F]
\end{aligned}
$$

In order to keep number of particles unchanged, we require $\langle N\rangle_s - \langle N\rangle_n = 0$, so we obtain:

$$
8\mu\Delta k_F + 4\hbar\omega_D\Delta k_F - \frac{2}{3}(\mu_1\epsilon_1 + \mu_2\epsilon_2)k_F = 0
$$

or

$$
\begin{aligned}
\frac{\Delta k_F}{k_F} &= \frac{1}{12}\frac{(\mu_1\epsilon_1+\mu_2\epsilon_2)}{\mu+\frac{1}{2}\hbar\omega_D} \\
&= \frac{1}{12}\frac{(\mu_1\epsilon_1+\mu_2\epsilon_2)}{\mu} \\
&= \frac{1}{12}(\epsilon_1+\epsilon_2)
\end{aligned}
$$

$$\begin{aligned}\mathscr{H} - \mu N &= \sum_{\vec{k}}[\frac{\hbar^2 k^2}{2m_1} - \mu_1']c^\dagger_{k\uparrow}c_{k\uparrow} \\ &+ \sum_{-\vec{k}}[\frac{\hbar^2 k^2}{2m_2} - \mu_2']c^\dagger_{-k\downarrow}c_{-k\downarrow} - \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime}c^\dagger_{k\prime\uparrow}c^\dagger_{-k\prime\downarrow}c_{-k\downarrow}c_{k\uparrow}\end{aligned} \quad (A.34)$$

and the test state function:

$$|\Psi\rangle = \prod_{D,\vec{k}}(u_k + v_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}) \prod_{D_1,\vec{k}} c^\dagger_{k\uparrow} \prod_{D_2,\vec{k}} c^\dagger_{-k\downarrow}|vac\rangle \quad (A.35)$$

and

$$\begin{aligned}\sum_{\vec{k}}\langle\Psi|c^\dagger_{k\uparrow}c_{k\uparrow}|\Psi\rangle &= \sum_{D,\vec{k}}\langle vac|(u_k^* + v_k^* c_{-k\downarrow}c_{k\uparrow})c^\dagger_{k\uparrow}c_{k\uparrow}(u_k + v_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow})|vac\rangle \\ &+ \sum_{D_1,\vec{k}}\langle vac|c_{k\uparrow}c^\dagger_{k\uparrow}c_{k\uparrow}c^\dagger_{k\uparrow}|vac\rangle \\ &= \sum_{D,\vec{k}} v_k^2 + \sum_{D_1,\vec{k}} 1 \\ \sum_{-\vec{k}}\langle\Psi|c^\dagger_{-k\downarrow}c_{-k\downarrow}|\Psi\rangle &= \sum_{D,-\vec{k}}\langle vac|(u_k^* + v_k^* c_{-k\downarrow}c_{k\uparrow})c^\dagger_{-k\downarrow}c_{-k\downarrow}(u_k + v_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow})|vac\rangle \\ &+ \sum_{D_2,-\vec{k}}\langle vac|c_{-k\downarrow}c^\dagger_{-k\downarrow}c_{-k\downarrow}c^\dagger_{-k\downarrow}|vac\rangle \\ &= \sum_{D,-\vec{k}} v_k^2 + \sum_{D_2,-\vec{k}} 1\end{aligned} \quad (A.36)$$

$$
\begin{aligned}
& \sum_{\vec{k},\vec{k\prime}} V_{k,k\prime} \langle \Psi | c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} | \Psi \rangle \\
= & \sum_{D,\vec{k},\vec{k'}} V_{k,k\prime} \langle vac | (u^*_k + v^*_k c_{-k\downarrow} c_{k\uparrow})(u^*_{k'} + v^*_{k'} c_{-k'\downarrow} c_{k'\uparrow}) c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} \\
& (u_{k'} + v_{k'} c^{\dagger}_{k'\uparrow} c^{\dagger}_{-k'\downarrow})(u_k + v_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow}) | vac \rangle \\
& + \sum_{D_1,\vec{k},\vec{k'}} V_{k,k\prime} \langle vac | c_{k\uparrow} c_{k'\uparrow} c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} c^{\dagger}_{k'\uparrow} c^{\dagger}_{k\uparrow} | vac \rangle \\
& + \sum_{D_2,\vec{k},\vec{k'}} V_{k,k\prime} \langle vac | c_{-k\downarrow} c_{-k'\downarrow} c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} c^{\dagger}_{-k'\downarrow} c^{\dagger}_{-k\downarrow} | vac \rangle \\
& + \sum_{D,\vec{k}} \sum_{D_1,\vec{k'}} V_{k,k\prime} \langle vac | (u^*_k + v^*_k c_{-k\downarrow} c_{k\uparrow}) c_{k'\uparrow} c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} c^{\dagger}_{k'\uparrow} \\
& (u_k + v_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow}) | vac \rangle \\
& + \sum_{D,\vec{k'}} \sum_{D_1,\vec{k}} V_{k,k\prime} \langle vac | (u^*_{k'} + v^*_{k'} c_{-k'\downarrow} c_{k'\uparrow}) c_{k\uparrow} c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} c^{\dagger}_{k\uparrow} \\
& (u_{k'} + v_{k'} c^{\dagger}_{k'\uparrow} c^{\dagger}_{-k'\downarrow}) | vac \rangle \\
& + \sum_{D,\vec{k}} \sum_{D_2,\vec{k'}} V_{k,k\prime} \langle vac | (u^*_k + v^*_k c_{-k\downarrow} c_{k\uparrow}) c_{-k'\downarrow} c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} c^{\dagger}_{-k'\downarrow} \\
& (u_k + v_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow}) | vac \rangle \\
& + \sum_{D,\vec{k'}} \sum_{D_2,\vec{k}} V_{k,k\prime} \langle vac | (u^*_{k'} + v^*_{k'} c_{-k'\downarrow} c_{k'\uparrow}) c_{-k\downarrow} c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} c^{\dagger}_{-k\downarrow} \\
& (u_{k'} + v_{k'} c^{\dagger}_{k'\uparrow} c^{\dagger}_{-k'\downarrow}) | vac \rangle \\
& + \sum_{D_1,\vec{k}} \sum_{D_2,\vec{k'}} V_{k,k\prime} \langle vac | c_{k\uparrow} c_{-k'\downarrow} c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} c^{\dagger}_{-k'\downarrow} c^{\dagger}_{k\uparrow} | vac \rangle \\
& + \sum_{D_1,\vec{k'}} \sum_{D_2,\vec{k}} V_{k,k\prime} \langle vac | c_{k'\uparrow} c_{-k\downarrow} c^{\dagger}_{k\prime\uparrow} c^{\dagger}_{-k\prime\downarrow} c_{-k\downarrow} c_{k\uparrow} c^{\dagger}_{-k\downarrow} c^{\dagger}_{k'\uparrow} | vac \rangle \\
= & \sum_{D,\vec{k},\vec{k'}} V_{k,k\prime} u^*_k v^*_{k\prime} v_k u_{k\prime}
\end{aligned}
\tag{A.37}
$$

After plugging the above equations into the trial Hamiltonian and keep this in mind: only region $D_{1\alpha}$ and $D_{2\beta}$ depair, and we write the sum and integral over region $D_{1\alpha} + D_{2\beta}$ as $D_1 + D_2$ for simplicity.

$$\begin{aligned}
& \langle\Psi|\mathscr{H} - \mu N|\Psi\rangle \\
= \; & \sum_{D,\vec{k}} (\frac{\hbar^2 k^2}{2m} - \mu_1')|v_k|^2 + \sum_{D_1,\vec{k}} (\frac{\hbar^2 k^2}{2m} - \mu_1') \\
& + \sum_{D,\vec{k}} (\frac{\hbar^2 k^2}{2m} - \mu_2')|v_k|^2 + \sum_{D_2,\vec{k}} (\frac{\hbar^2 k^2}{2m} - \mu_2') - \sum_{D,\vec{k},\vec{k}'} V_{k,k\prime} u_k^* v_{k\prime}^* v_k u_{k\prime} \\
= \; & \sum_{D,\vec{k}} 2(\frac{\hbar^2 k^2}{2m} - \frac{\mu_1' + \mu_2'}{2})|v_k|^2 + \sum_{D_1,\vec{k}} (\frac{\hbar^2 k^2}{2m} - \frac{\mu_1' + \mu_2'}{2} - \frac{\mu_1' - \mu_2'}{2}) \\
& + \sum_{D_2,\vec{k}} (\frac{\hbar^2 k^2}{2m} - \frac{\mu_1' + \mu_2'}{2} + \frac{\mu_1' - \mu_2'}{2}) - \sum_{D,\vec{k},\vec{k}'} V_{k,k\prime} u_k^* v_{k\prime}^* v_k u_{k\prime} \\
= \; & \sum_{D,\vec{k}} 2\epsilon_k'|v_k|^2 + \sum_{D_1,\vec{k}} (\epsilon_k' - \delta\mu') + \sum_{D_2,\vec{k}} (\epsilon_k' + \delta\mu') \\
& - \sum_{D,\vec{k},\vec{k}'} V_{k,k\prime} u_k^* v_{k\prime}^* v_k u_{k\prime} \\
= \; & \sum_{\vec{k}} 2\epsilon_k'|v_k|^2 - \sum_{D_1+D_2\vec{k}} 2\epsilon_k'|v_k|^2 + \sum_{D_1,\vec{k}} (\epsilon_k' - \delta\mu') + \sum_{D_2,\vec{k}} (\epsilon_k' + \delta\mu') \\
& - \sum_{D,\vec{k},\vec{k}'} V_{k,k\prime} u_k^* v_{k\prime}^* v_k u_{k\prime} \\
= \; & \sum_{\vec{k}} 2\epsilon_k'|v_k|^2 - \sum_{D_1+D_2\vec{k}} \epsilon_k'(2|v_k|^2 - 1) - \sum_{D_1,\vec{k}} \delta\mu' + \sum_{D_2,\vec{k}} \delta\mu' \\
& - \sum_{D,\vec{k},\vec{k}'} V_{k,k\prime} u_k^* v_{k\prime}^* v_k u_{k\prime}
\end{aligned}$$

For S-wave case and real gap we obtain:

$$
\begin{aligned}
&\langle\Psi|\mathscr{H}-\mu N|\Psi\rangle_s \\
=\ & \sum_{\vec{k}} 2\epsilon'_k|v_k|^2 - \sum_{D_1+D_2\vec{k}} \epsilon'_k(2|v_k|^2-1) - \sum_{D_1,\vec{k}} \delta\mu' + \sum_{D_2,\vec{k}} \delta\mu' \\
& -V\left(\sum_{D,\vec{k}} u_k^* v_k\right)^2 \\
=\ & \frac{V}{(2\pi)^3}\iiint 2\epsilon'_k|v_k|^2 k^2 \sin\theta \,\mathrm{d}k\,\mathrm{d}\theta\,\phi \\
& -\frac{V}{(2\pi)^3}\iiint\limits_{D_1+D_2\vec{k}} \epsilon'_k(2|v_k|^2-1)k^2\sin\theta\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi \\
& -\frac{V}{(2\pi)^3}\iiint\limits_{D_1} \delta\mu' k^2\sin\theta\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi + \frac{V}{(2\pi)^3}\iiint\limits_{D_2} \delta\mu' k^2\sin\theta\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi \\
& -V\left(\frac{V}{(2\pi)^3}\iiint\limits_{D} u_k^* v_k k^2 \sin\theta\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi\right)^2 \\
=\ & \frac{N(0)}{2}\iint 2\epsilon'_k|v_k|^2\,\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta \\
& -\frac{N(0)}{2}\iint\limits_{D_1+D_2\vec{k}} \epsilon'_k(2|v_k|^2-1)\,\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta \\
& -\frac{N(0)}{2}\iint\limits_{D_1}\delta\mu'\,\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta + \frac{N(0)}{2}\iint\limits_{D_2}\delta\mu'\,\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta \\
& -V\left(\frac{N(0)}{2}\iint\limits_{D} u_k^* v_k\,\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta\right)^2 \\
=\ & \frac{N(0)}{2}\iint\left(\epsilon'_k - \frac{\epsilon'_k(\epsilon_k+\alpha)}{\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}}\right)\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta \\
& +\frac{N(0)}{2}\iint\limits_{D_1+D_2\vec{k}} \epsilon'_k\frac{\epsilon_k+\alpha}{\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}}\,\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta \\
& -\frac{N(0)}{2}\iint\limits_{D_1}\delta\mu'\,\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta + \frac{N(0)}{2}\iint\limits_{D_2}\delta\mu'\,\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta \\
& -V\left(N(0)\iint\limits_{D}\left(\frac{1}{2}\frac{\Delta}{\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}}\right)\mathrm{d}\epsilon_k\,\mathrm{d}\cos\theta\right)^2
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{N(0)}{2} \iint \left(\epsilon_k - \frac{\epsilon_k(\epsilon_k + \alpha)}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} \right) \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \\
& + \frac{N(0)}{2} \iint \left(\Delta\mu - \frac{\Delta\mu(\epsilon_k + \alpha)}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} \right) \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \\
& + \frac{N(0)}{2} \iint\limits_{D_1 + D_2 \vec{k}} \epsilon_k \frac{\epsilon_k + \alpha}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} \, \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \\
& + \frac{N(0)}{2} \iint\limits_{D_1 + D_2 \vec{k}} \Delta\mu \frac{\epsilon_k + \alpha}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} \, \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \\
& - \frac{N(0)}{2} \iint\limits_{D_1} \delta\mu' \, \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta + \frac{N(0)}{2} \iint\limits_{D_2} \delta\mu' \, \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \\
& - V \left(N(0) \iint\limits_{D} \left(\frac{1}{2} \frac{\Delta}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}}\right) \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \right)^2 \\
= \; & \frac{N(0)}{2} \iint \left(\epsilon_k - \sqrt{(\epsilon_k + \alpha)^2 + \Delta^2} + \frac{\Delta^2}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} \right. \\
& \left. + \frac{\alpha(\epsilon_k + \alpha)}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} + \Delta\mu - \frac{\Delta\mu(\epsilon_k + \alpha)}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} \right) \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \\
& + \frac{N(0)}{2} \iint\limits_{D_1 + D_2 \vec{k}} \left(\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2} - \frac{\Delta^2}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} \right. \\
& \left. - \frac{\alpha(\epsilon_k + \alpha)}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} + \frac{\Delta\mu(\epsilon_k + \alpha)}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}} \right) \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \\
& - \frac{N(0)}{2} \iint\limits_{D_1} \delta\mu' \, \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta + \frac{N(0)}{2} \iint\limits_{D_2} \delta\mu' \, \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \\
& - V \left(\frac{N(0)}{2} \iint\limits_{D} \left(\frac{1}{2} \frac{\Delta}{\sqrt{(\epsilon_k + \alpha)^2 + \Delta^2}}\right) \mathrm{d}\epsilon_k \, \mathrm{d}\cos\theta \right)^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{N(0)}{2}\int\Big(\frac{1}{2}\epsilon_k^2-\frac{1}{2}\Delta^2 ln\big(\epsilon_k+\alpha+\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\big)\\
& -\frac{1}{2}(\epsilon_k+\alpha)\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}+\Delta\mu\epsilon_k-\Delta\mu\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\\
& +\alpha\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}+\Delta^2 ln\big(\epsilon_k+\alpha+\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\big)\Big)\Big|_{-\hbar\omega_D}^{\hbar\omega_D}\,\mathrm{d}\cos\theta\\
& +\frac{N(0)}{2}\int\limits_{D_1+D_2}\Big(\frac{1}{2}\Delta^2 ln\big(\epsilon_k+\alpha+\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\big)\\
& +\frac{1}{2}(\epsilon_k+\alpha)\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}+\Delta\mu\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\\
& -\alpha\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}-\Delta^2 ln\big(\epsilon_k+\alpha+\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\big)\Big)\Big|_{\epsilon_-}^{\epsilon_+}\,\mathrm{d}\cos\theta\\
& -\frac{N(0)}{2}\int\limits_{D_1}\delta\mu'\epsilon_k\big|_{\epsilon_-}^{\epsilon_+}\,\mathrm{d}\cos\theta+\frac{N(0)}{2}\int\limits_{D_2}\delta\mu'\epsilon_k\big|_{\epsilon_-}^{\epsilon_+}\,\mathrm{d}\cos\theta\\
& -V\Big(\frac{N(0)}{2}\int\frac{1}{2}\Delta ln\big(\epsilon_k+\alpha+\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\big)\big|_{-\hbar\omega_D}^{\hbar\omega_D}\,\mathrm{d}\cos\theta\\
& -\frac{N(0)}{2}\int\limits_{D_1+D_2}\frac{1}{2}\Delta ln\big(\epsilon_k+\alpha+\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\big)\big|_{\epsilon_-}^{\epsilon_+}\,\mathrm{d}\cos\theta\Big)^2\\
= \quad & \frac{N(0)}{2}\int\Big(\frac{1}{2}\Delta^2 ln\big(\epsilon_k+\alpha+\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\big)\\
& -\frac{1}{2}(\epsilon_k-\alpha)\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}+\Delta\mu\epsilon_k-\Delta\mu\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\Big)\Big|_{-\hbar\omega_D}^{\hbar\omega_D}\,\mathrm{d}\cos\theta\\
& +\frac{N(0)}{2}\int\limits_{D_1+D_2}\Big(-\frac{1}{2}\Delta^2 ln\big(\epsilon_k+\alpha+\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\big)\\
& +\frac{1}{2}(\epsilon_k-\alpha)\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}+\Delta\mu\sqrt{(\epsilon_k+\alpha)^2+\Delta^2}\Big)\Big|_{\epsilon_-}^{\epsilon_+}\,\mathrm{d}\cos\theta\\
& -\frac{N(0)}{2}\int\limits_{D_1}2\delta\mu'\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\,\mathrm{d}\cos\theta\\
& +\frac{N(0)}{2}\int\limits_{D_2}2\delta\mu'\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\,\mathrm{d}\cos\theta\\
& -V\Big(\frac{N(0)}{2}\int\frac{1}{2}\Delta ln\big(\frac{\hbar\omega_D+\alpha+\sqrt{(\hbar\omega_D+\alpha)^2+\Delta^2}}{-\hbar\omega_D+\alpha+\sqrt{(-\hbar\omega_D+\alpha)^2+\Delta^2}}\big)\,\mathrm{d}\cos\theta\\
& -\frac{N(0)}{2}\int\limits_{D_1+D_2}\frac{1}{2}\Delta ln\big(\frac{\epsilon_++\alpha+\sqrt{(\epsilon_++\alpha)^2+\Delta^2}}{\epsilon_-+\alpha+\sqrt{(\epsilon_-+\alpha)^2+\Delta^2}}\big)\,\mathrm{d}\cos\theta\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{N(0)}{2}\int\Bigg(\frac{1}{2}\Delta^2 ln(\frac{\hbar\omega_D+\alpha+\sqrt{(\hbar\omega_D+\alpha)^2+\Delta^2}}{-\hbar\omega_D+\alpha+\sqrt{(-\hbar\omega_D+\alpha)^2+\Delta^2}}) \\
& -\frac{1}{2}(\hbar\omega_D-\alpha)(\hbar\omega_D+\alpha)\sqrt{1+\frac{\Delta^2}{(\hbar\omega_D+\alpha)^2}} \\
& +\frac{1}{2}(-\hbar\omega_D-\alpha)(\hbar\omega_D-\alpha)\sqrt{1+\frac{\Delta^2}{(-\hbar\omega_D+\alpha)^2}} \\
& +2\Delta\mu\hbar\omega_D-\Delta\mu\sqrt{(\hbar\omega_D+\alpha)^2+\Delta^2}+\Delta\mu\sqrt{(\hbar\omega_D-\alpha)^2+\Delta^2}\Bigg)\,\mathrm{d}\cos\theta \\
& +\frac{N(0)}{2}\int\limits_{D_1+D_2}\Bigg(-\frac{1}{2}\Delta^2 ln(\frac{\epsilon_+ +\alpha+\sqrt{(\epsilon_+ +\alpha)^2+\Delta^2}}{\epsilon_- +\alpha+\sqrt{(\epsilon_- +\alpha)^2+\Delta^2}}) \\
& +\frac{1}{2}\big(-2\alpha+\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\big)|\delta\mu-z\cos^2\theta| \\
& -\frac{1}{2}\big(-2\alpha-\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\big)|\delta\mu-z\cos^2\theta| \\
& +\Delta\mu|\delta\mu-z\cos^2\theta|-\Delta\mu|\delta\mu-z\cos^2\theta| \\
& -\frac{N(0)}{2}\int\limits_{D_1}2\delta\mu'\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\,\mathrm{d}\cos\theta \\
& +\frac{N(0)}{2}\int\limits_{D_2}2\delta\mu'\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\,\mathrm{d}\cos\theta \\
& -V\Bigg(\frac{N(0)}{2}\int\frac{1}{2}\Delta ln(\frac{4\hbar^2\omega_D^2-4\alpha^2}{\Delta^2})\,\mathrm{d}\cos\theta \\
& -\frac{N(0)}{2}\int\limits_{D_1+D_2}\Delta ln(\frac{|\delta\mu-z\cos^2\theta|+\sqrt{(\delta\mu-z\cos^2\theta\,)^2-\Delta^2}}{\Delta})\,\mathrm{d}\cos\theta\Bigg)^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{N(0)}{2}\int\Bigg(\frac{1}{2}\Delta^2 ln(\frac{4\hbar^2\omega_D^2-4\alpha(\theta)^2}{\Delta^2})-\hbar^2\omega_D^2+\alpha(\theta)^2-\frac{\Delta^2}{2} \\
& +2\Delta\mu\hbar\omega_D-2\Delta\mu\alpha(\theta)\Bigg)\,\mathrm{d}\cos\theta \\
& +\frac{N(0)}{2}\int\limits_{D_1+D_2}\Bigg(-\Delta^2 ln(\frac{|\delta\mu-z\cos^2\theta|+\sqrt{(\delta\mu-z\cos^2\theta\,)^2-\Delta^2}}{\Delta}) \\
& +|\delta\mu-z\cos^2\theta|\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\Bigg)\,\mathrm{d}\cos\theta \\
& -\frac{N(0)}{2}\int\limits_{D_1}2\delta\mu'\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\,\mathrm{d}\cos\theta \\
& +\frac{N(0)}{2}\int\limits_{D_2}2\delta\mu'\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\,\mathrm{d}\cos\theta \\
& -V\Bigg(\frac{N(0)}{2}\int\frac{1}{2}\Delta ln(\frac{4\hbar^2\omega_D^2-4\alpha^2}{\Delta^2})\,\mathrm{d}\cos\theta \\
& -\frac{N(0)}{2}\int\limits_{D_1+D_2}\Delta ln(\frac{|\delta\mu-z\cos^2\theta|+\sqrt{(\delta\mu-z\cos^2\theta\,)^2-\Delta^2}}{\Delta})\,\mathrm{d}\cos\theta\Bigg)^2 \\
= \quad & N(0)\Bigg(\Delta^2 ln\frac{2\hbar\omega_D}{\Delta}-\hbar^2\omega_D^2+\frac{1}{5}(\mu_1\epsilon_1+\mu_2\epsilon_2)^2-\frac{\Delta^2}{2}+2\Delta\mu\hbar\omega_D \\
& -\frac{2}{3}\Delta\mu(\mu_1\epsilon_1+\mu_2\epsilon_2)\Bigg) \\
& +\frac{N(0)}{2}\int\limits_{D_1+D_2}\Bigg(-\Delta^2 ln(\frac{|\delta\mu-z\cos^2\theta|+\sqrt{(\delta\mu-z\cos^2\theta\,)^2-\Delta^2}}{\Delta}) \\
& +|\delta\mu-z\cos^2\theta|\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\Bigg)\,\mathrm{d}\cos\theta \\
& -\frac{N(0)}{2}\int\limits_{D_1}2\delta\mu'\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\,\mathrm{d}\cos\theta \\
& +\frac{N(0)}{2}\int\limits_{D_2}2\delta\mu'\sqrt{(\delta\mu-z\cos^2\theta)^2-\Delta^2}\,\mathrm{d}\cos\theta \\
& -V\Bigg(N(0)\Delta ln\frac{2\hbar\omega_D}{\Delta} \\
& -\frac{N(0)}{2}\int\limits_{D_1+D_2}\Delta ln(\frac{|\delta\mu-z\cos^2\theta|+\sqrt{(\delta\mu-z\cos^2\theta\,)^2-\Delta^2}}{\Delta})\,\mathrm{d}\cos\theta\Bigg)^2
\end{aligned}
$$

And for $E_n(0)$, we only consider the particles within $(\mu - \hbar\omega_D, \mu + \hbar\omega_D)$:

$$
\begin{aligned}
E_n(0) &= \sum_{\vec{k}} \epsilon'_k - \sum_{\vec{k}} |\epsilon'_k| \\
&= N(0)\int_{\mu-\hbar\omega_D-\mu'}^{\mu+\hbar\omega_D-\mu'} \epsilon'_k \,\mathrm{d}\epsilon'_k - N(0)\int_{\mu-\hbar\omega_D-\mu'}^{\mu+\hbar\omega_D-\mu'} |\epsilon'_k| \,\mathrm{d}\epsilon'_k \\
&= \frac{N(0)}{2}\epsilon'^2_k\Big|_{\Delta\mu-\hbar\omega_D}^{\Delta\mu+\hbar\omega_D} - \frac{N(0)}{2}\epsilon'^2_k\Big|_{0}^{\Delta\mu+\hbar\omega_D} - \frac{N(0)}{2}\epsilon'^2_k\Big|_{0}^{\Delta\mu-\hbar\omega_D} \\
&= -N(0)\epsilon'^2_k\Big|_{0}^{\Delta\mu-\hbar\omega_D} \\
&= -N(0)(\Delta\mu - \hbar\omega_D)^2 \\
E_{Pauli}(0) &= \sum_{\vec{k}} \epsilon'_k - \sum_{\epsilon_k>|\delta\mu'|} |\epsilon'_k| - \sum_{\epsilon_k<|\delta\mu'|} \delta\mu \\
&= N(0)\int_{\mu-\hbar\omega_D-\mu'}^{\mu+\hbar\omega_D-\mu'} \epsilon'_k \,\mathrm{d}\epsilon'_k - N(0)\int_{\delta\mu'}^{\mu+\hbar\omega_D-\mu'} |\epsilon'_k| \,\mathrm{d}\epsilon'_k \\
&\quad -N(0)\int_{\mu-\hbar\omega_D-\mu'}^{-\delta\mu'} |\epsilon'_k| \,\mathrm{d}\epsilon'_k - N(0)\int_{-\delta\mu'}^{\delta\mu'} \delta\mu \,\mathrm{d}\epsilon'_k \\
&= \frac{N(0)}{2}\epsilon'^2_k\Big|_{\Delta\mu-\hbar\omega_D}^{\Delta\mu+\hbar\omega_D} - \frac{N(0)}{2}\epsilon'^2_k\Big|_{\delta\mu}^{\Delta\mu+\hbar\omega_D} - \frac{N(0)}{2}\epsilon'^2_k\Big|_{-\delta\mu}^{\Delta\mu-\hbar\omega_D} \\
&\quad -2N(0)\delta\mu\epsilon'_k\Big|_{0}^{\delta\mu'} \\
&= -N(0)(\Delta\mu - \hbar\omega_D)^2 + \frac{N(0)}{2}\delta\mu'^2 - 2N(0)\delta\mu'^2 \\
&= -N(0)(\Delta\mu - \hbar\omega_D)^2 - N(0)\delta\mu'^2
\end{aligned}
\tag{A.38}
$$

where

$$\begin{aligned}
\Delta\mu &= \mu - \mu' \\
&= \frac{\hbar^2 k_F^2}{2m} - \frac{\hbar^2 k_F'^2}{2m} \\
&= \frac{\hbar^2}{2m}(k_F + k_F')\Delta k_F \\
&\simeq 2\frac{\hbar^2}{2m}k_F^2 \frac{\Delta k_F}{k_F} \\
&= \frac{1}{6}(\mu_1\epsilon_1 + \mu_2\epsilon_2)
\end{aligned} \tag{A.39}$$

Then we let $x = \cos\theta$ and calculate partial derivatives:

$$
\begin{aligned}
& E_s(0) - E_n(0) \\
= \; & N(0)\Bigg(\Delta^2 ln\frac{2\hbar\omega_D}{\Delta} + \frac{1}{5}(\mu_1\epsilon_1 + \mu_2\epsilon_2)^2 - \frac{\Delta^2}{2} - \frac{1}{9}(\mu_1\epsilon_1 + \mu_2\epsilon_2)^2 \\
& + \frac{1}{36}(\mu_1\epsilon_1 + \mu_2\epsilon_2)^2\Bigg) \\
& + \frac{N(0)}{2}\int\limits_{D_1+D_2}\Bigg(- \Delta^2 ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta}) \\
& + |\delta\mu - zx^2|\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\Bigg)\,\mathrm{d}x \\
& - \frac{N(0)}{2}\int\limits_{D_1} 2\delta\mu'\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x \\
& + \frac{N(0)}{2}\int\limits_{D_2} 2\delta\mu'\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x \\
& - V\Bigg(N(0)\Delta ln\frac{2\hbar\omega_D}{\Delta} \\
& - \frac{N(0)}{2}\int\limits_{D_1+D_2}\Delta ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Bigg)^2 \\
= \; & N(0)\Bigg(\Delta^2 ln\frac{2\hbar\omega_D}{\Delta} + \frac{7}{60}(\mu_1\epsilon_1 + \mu_2\epsilon_2)^2 - \frac{\Delta^2}{2}\Bigg) \\
& + \frac{N(0)}{2}\int\limits_{D_1+D_2}\Bigg(- \Delta^2 ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta}) \\
& + |\delta\mu - zx^2|\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\Bigg)\,\mathrm{d}x \\
& - \frac{N(0)}{2}\int\limits_{D_1} 2\delta\mu'\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x \\
& + \frac{N(0)}{2}\int\limits_{D_2} 2\delta\mu'\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x \\
& - V\Bigg(N(0)\Delta ln\frac{2\hbar\omega_D}{\Delta} \\
& - \frac{N(0)}{2}\int\limits_{D_1+D_2}\Delta ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Bigg)^2
\end{aligned}
\tag{A.40}
$$

$$
\begin{aligned}
&\frac{\partial(E_s - E_n(0))}{\partial\Delta} \\
= \quad & 2N(0)\Delta(ln\frac{2\hbar\omega_D}{\Delta} - 1) - N(0)\Delta \int\limits_{D_1+D_2} ln\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\,\mathrm{d}x \\
& + N(0)\Delta\int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x - N(0)\Delta\int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x \\
& -2VN(0)^2\Delta\Big(ln\frac{2\hbar\omega_D}{\Delta} - \frac{1}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\Big) \\
& \times\Big(ln\frac{2\hbar\omega_D}{\Delta} - 1 - \frac{1}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - z\cos^2\theta\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x \\
& + \frac{1}{2}\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x\Big) = 0 \qquad \text{(A.41)}
\end{aligned}
$$

$$
\begin{aligned}
& 2ln\frac{2\hbar\omega_D}{\Delta} - 2 - \int\limits_{D_1+D_2} ln\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\,\mathrm{d}\cos\theta \\
& + \int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}\cos\theta - \int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x \\
& -\Big(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\Big) \\
& \times\Big(2ln\frac{2\hbar\omega_D}{\Delta} - 2 - \int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - z\cos^2\theta| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x \\
& + \int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x\Big) = 0
\end{aligned}
$$

(A.42)

$$
\begin{aligned}
&\int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x - \int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x \\
&-\Bigg(VN(0)ln\frac{2\hbar\omega_D}{\Delta} \\
&-\frac{VN(0)}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x - 1\Bigg) \\
&\times\Bigg(2ln\frac{2\hbar\omega_D}{\Delta} - 2 - \int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\Bigg) \\
&-\Bigg(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\Bigg) \\
&\times\Bigg(\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x\Bigg) = 0
\end{aligned}
\tag{A.43}
$$

$$
\begin{aligned}
&\int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x - \int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x \\
&-\left(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 1\right) \times \left(2ln\frac{2\hbar\omega_D}{\Delta} - 2\right) \\
&+\left(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 1\right) \times \int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x \\
&+\frac{VN(0)}{2} \int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x \times \left(2ln\frac{2\hbar\omega_D}{\Delta} - 2\right) \\
&-\frac{VN(0)}{2}\left(\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\right)^2 \\
&-\left(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\right) \\
&\times\left(\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x\right) = 0
\end{aligned}
\tag{A.44}
$$

$$
\begin{aligned}
&\int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x - \int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x \\
&-\Big(2VN(0)(ln\frac{2\hbar\omega_D}{\Delta})^2 - 2VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 2ln\frac{2\hbar\omega_D}{\Delta} + 2\Big) \\
&+\Big(2VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 1 - VN(0)\Big) \\
&\times \int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta})\,\mathrm{d}x \\
&-\frac{VN(0)}{2}\Big(\int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Big)^2 \\
&-\Big(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)}{2}\int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Big) \\
&\times\Big(\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x\Big) = 0
\end{aligned}
\tag{A.45}
$$

$$
\begin{aligned}
&\int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x - \int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x \\
&-\Big(2VN(0)(ln\frac{2\hbar\omega_D}{\Delta})^2 - 2VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 2ln\frac{2\hbar\omega_D}{\Delta} + 2\Big) \\
&+\Big(2VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 1 - VN(0)\Big) \\
&\times \int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta})\,\mathrm{d}x \\
&-\frac{VN(0)}{2}\Big(\int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta})\,\mathrm{d}\cos\theta\Big)^2 \\
&-\Big(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)}{2}\int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Big) \\
&\times\Big(\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x\Big) = 0
\end{aligned}
\tag{A.46}
$$

$$
\begin{aligned}
&\int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\, \mathrm{d}x - \int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\, \mathrm{d}x \\
&+\Bigg(2VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 1 - gN(0)\Bigg) \\
&\times \int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\, \mathrm{d}x \\
&-\frac{VN(0)}{2}\Bigg(\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\, \mathrm{d}x\Bigg)^2 \\
&-\Bigg(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\, \mathrm{d}x\Bigg) \\
&\times\Bigg(\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\, \mathrm{d}x\Bigg) = \\
&\Bigg(2VN(0)(ln\frac{2\hbar\omega_D}{\Delta})^2 - 2VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 2ln\frac{2\hbar\omega_D}{\Delta} + \frac{(VN(0)+1)^2}{2VN(0)} \\
&-\frac{(VN(0)+1)^2}{2VN(0)} + 2\Bigg)
\end{aligned}
\tag{A.47}
$$

$$
\begin{aligned}
&\int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x - \int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x \\
&+\Big(2VN(0)ln\frac{2\hbar\omega_D}{\Delta} - 1 - VN(0)\Big) \\
&\times \int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x \\
&-\frac{VN(0)}{2}\Big(\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\Big)^2 \\
&-\Big(VN(0)ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\Big) \\
&\times\Big(\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x\Big) + \frac{(VN(0)+1)^2}{2VN(0)} - 2 = \\
&\Big(\sqrt{2VN(0)}ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)+1}{\sqrt{2VN(0)}}\Big)^2
\end{aligned}
\tag{A.48}
$$

Let

$$
leftside = \Big(\sqrt{2VN(0)}ln\frac{2\hbar\omega_D}{\Delta} - \frac{VN(0)+1}{\sqrt{2VN(0)}}\Big)^2 \tag{A.49}
$$

Then we obtain:

$$
ln\frac{2\hbar\omega_D}{\Delta} = \frac{VN(0)+1}{2gN(0)} + \sqrt{\frac{leftside}{2VN(0)}} \tag{A.50}
$$

$$
\frac{1}{N(0)V} - ln\frac{\Delta}{\Delta_0} = \frac{VN(0)+1}{2gN(0)} + \sqrt{\frac{leftside}{2VN(0)}} \tag{A.51}
$$

$$\frac{\Delta}{\Delta_0} = exp\{\frac{1 - VN(0)}{2VN(0)} - \sqrt{\frac{leftside}{2VN(0)}}\} \tag{A.52}$$

in the unit of Δ_0 we have:

$$\begin{aligned} VN(0)ln\frac{2\hbar\omega_D}{\Delta} &= VN(0)ln\frac{2\hbar\omega_D}{\Delta_0} - VN(0)ln\frac{\Delta}{\Delta_0} \\ &= 1 - VN(0)ln\frac{\Delta}{\Delta_0} \\ &= 1 - VN(0)ln\Delta \end{aligned} \tag{A.53}$$

Then we obtain :

$$
\begin{aligned}
&\int\limits_{D_1} \frac{\delta\mu'/z}{\sqrt{(\delta\mu/z - x^2\ )^2 - \Delta^2/z^2}}\,\mathrm{d}x - \int\limits_{D_2} \frac{\delta\mu'/z}{\sqrt{(\delta\mu/z - x^2\ )^2 - \Delta^2/z^2}}\,\mathrm{d}x \\
&+\Big(1 - VN(0) - 2VN(0)ln\Delta\Big) \\
&\times \int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu/z - x^2| + \sqrt{(\delta\mu/z - x^2\ )^2 - \Delta^2/z^2}}{\Delta/z}\big)\,\mathrm{d}x \\
&-\frac{VN(0)}{2}\Big(\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu/z - x^2| + \sqrt{(\delta\mu/z - x^2\ )^2 - \Delta^2/z^2}}{\Delta/z}\big)\,\mathrm{d}x\Big)^2 \\
&-\Big(1 - VN(0)ln\Delta \\
&-\frac{VN(0)}{2}\int\limits_{D_1+D_2} ln\frac{|\delta\mu/z - x^2| + \sqrt{(\delta\mu/z - x^2\ )^2 - \Delta^2/z^2}}{\Delta/z}\,\mathrm{d}x\Big) \\
&\times\Big(\int\limits_{D_1+D_2} \frac{|\delta\mu/z - x^2|}{\sqrt{(\delta\mu/z - x^2\ )^2 - \Delta^2/z^2}}\,\mathrm{d}x\Big) + \frac{(VN(0)+1)^2}{2VN(0)} - 2 \\
&= \Big(\sqrt{2VN(0)}ln\frac{2\hbar\omega_D}{\Delta} - \frac{gN(0)+1}{\sqrt{2gN(0)}}\Big)^2 \\
&= \frac{1}{2VN(0)}\Big(2gN(0)ln\frac{2\hbar\omega_D}{\Delta} - VN(0) - 1\Big)^2 \\
&= \frac{1}{2VN(0)}\Big(1 - 2gN(0)ln\Delta - VN(0)\Big)^2
\end{aligned}
\tag{A.54}
$$

and

$$
\Delta = exp\big(\frac{1 - gN(0)}{2gN(0)} \pm \sqrt{\frac{leftside}{2gN(0)}}\big)
\tag{A.55}
$$

and

$$\begin{aligned}
&\frac{\partial(E_s - E_n(0))}{\partial\delta\mu} \\
= \;& N(0)\int\limits_{D_1}\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x - N(0)\int\limits_{D_2}\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}\,\mathrm{d}x \\
& -N(0)\int\limits_{D_1+D_2}\frac{\delta\mu'|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x \\
& +N(0)\Delta^2\Bigg(VN(0)ln\frac{2\hbar\omega_D}{\Delta} \\
& -\frac{VN(0)}{2}\int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Bigg) \\
& \times\Bigg(\int\limits_{D_1}\frac{1}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x - \int\limits_{D_2}\frac{1}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x\Bigg) \qquad \text{(A.56)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial(E_s - E_n(0))}{\partial z} \\
= \;& -N(0)\int\limits_{D_1}\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}x^2\,\mathrm{d}x + N(0)\int\limits_{D_2}\sqrt{(\delta\mu - zx^2)^2 - \Delta^2}x^2\,\mathrm{d}x \\
& +N(0)\int\limits_{D_1+D_2}\frac{\delta\mu'|\delta\mu - zx^2|x^2}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x \\
& +N(0)\Delta^2\Bigg(gN(0)ln\frac{2\hbar\omega_D}{\Delta} \\
& -\frac{VN(0)}{2}\int\limits_{D_1+D_2} ln(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}{\Delta})\,\mathrm{d}x\Bigg) \\
& \times\Bigg(-\int\limits_{D_1}\frac{x^2}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x + \int\limits_{D_2}\frac{x^2}{\sqrt{(\delta\mu - zx^2\,)^2 - \Delta^2}}\,\mathrm{d}x\Bigg) \qquad \text{(A.57)}
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{\partial^2(E_s - E_n(0))}{\partial^2 \Delta} \\
= \quad & 2N(0)(ln\frac{2\hbar\omega_D}{\Delta} - 2) - N(0)\int\limits_{D_1+D_2} ln\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\,\mathrm{d}x \\
& + N(0)\int\limits_{D_1+D_2} \frac{|\delta\mu/z - x^2|}{\sqrt{(\delta\mu/z - x^2\ )^2 - \Delta^2/z^2}}\,\mathrm{d}x \\
& + N(0)\int\limits_{D_1} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x - N(0)\int\limits_{D_2} \frac{\delta\mu'}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x \\
& + N(0)\int\limits_{D_1} \frac{\delta\mu'\Delta^2}{\big(\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}\big)^3}\,\mathrm{d}x - N(0)\int\limits_{D_2} \frac{\delta\mu'\Delta^2}{\big(\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}\big)^3}\,\mathrm{d}x \\
& - 2VN(0)^2\Bigg(ln\frac{2\hbar\omega_D}{\Delta} - 1 - \frac{1}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x \\
& + \frac{1}{2}\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|}{\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}\,\mathrm{d}x\Bigg)^2 \\
& - 2VN(0)^2\Bigg(- 1 + \frac{1}{2}\int\limits_{D_1+D_2} \frac{|\delta\mu - zx^2|^3}{\big(\sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}\big)^3}\,\mathrm{d}x\Bigg) \\
& \times\Bigg(ln\frac{2\hbar\omega_D}{\Delta} - \frac{1}{2}\int\limits_{D_1+D_2} ln\big(\frac{|\delta\mu - zx^2| + \sqrt{(\delta\mu - zx^2\ )^2 - \Delta^2}}{\Delta}\big)\,\mathrm{d}x\Bigg) \qquad \text{(A.58)}
\end{aligned}
$$

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